THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and fifty-seventh regular meeting of the Society was held in New York City on Saturday, February 24, 1912. The attendance at the two sessions included the following forty-one members:

Dr. F. W. Beal, Mr. A. A. Bennett, Professor W. J. Berry, Professor G. D. Birkhoff, Professor Joseph Bowden, Professor E. W. Brown, Professor A. B. Coble, Professor F. N. Cole, Dr. Elizabeth B. Cowley, Dr. H. B. Curtis, Dr. L. L. Dines, Mr. E. P. R. Duval, Professor H. B. Fine, Professor T. S. Fiske, Professor W. B. Fite, Professor C. C. Grove, Professor H. E. Hawkes, Dr. Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Professor J. H. Maclagan-Wedderburn, Dr. E. J. Miles, Dr. H. H. Mitchell, Mr. F. S. Nowlan, Professor W. F. Osgood, Mr. E. S. Palmer, Dr. H. W. Reddick, Dr. J. E. Rowe, Mr. L. P. Siceloff, Professor D. E. Smith, Professor P. F. Smith, Professor Virgil Snyder, Professor Henry Taber, Professor H. D. Thompson, Dr. M. O. Tripp, Professor Oswald Veblen, Mr. H. E. Webb, Professor H. S. White, Professor A. H. Wilson, Professor J. W. Young.

The President of the Society, Professor Fine, occupied the chair. The Council announced the election of the following persons to membership in the Society: Mr. J. W. Alexander, Princeton University; Mr. A. A. Bennett, Princeton University; Professor J. G. Coffin, College of the City of New York; Professor G. H. Cresse, Middlebury College; Mr. C. R. Dines, Dartmouth College; Professor H. E. Jordan, University of Kansas; Mr. F. S. Nowlan, Columbia University; Professor C. W. Watkeys, University of Rochester. Eight applications for membership in the Society were received.

Announcement was made of the recent gift by Dr. Emory McClintock, second President of the Society, of over four hundred valuable mathematical books to the Society's Library. The gift also includes a large number of pamphlets and reprints of important mathematical papers. The grateful acknowledgments of the Society were tendered to Dr. McClintock by resolution.

The following papers were read at this meeting:

(1) Mr. S. A. JOFFE: "Sums of like powers of natural numbers."

(2) Professor G. A. MILLER: "Second note on the groups generated by operators transforming each other into their inverses."

(3) Dr. S. LEFSCHETZ: "On remarkable points of curves."

(4) Dr. S. E. URNER: "Certain singularities of point transformations in space of three dimensions."

(5) Professor A. B. COBLE: "The characteristic theory of the odd and even theta functions as related to finite geometry."

(6) Dr. H. H. MITCHELL: "Some quaternary groups with particular prime moduli."

(7) Dr. J. E. ROWE: "The undulation and cusp invariants of the R^n."

(8) Professor W. F. OSGOOD: "A necessary and sufficient condition that a single-valued function in a projective space be rational."

(9) Dr. DUNHAM JACKSON: "On the convergence of the development of a continuous function according to Legendre's polynomials."

(10) Mr. K. P. WILLIAMS: "The solutions of non-homogeneous linear difference equations and their asymptotic forms."

(11) Dr. E. J. MILES: "Note on the isoperimetric problem with discontinuous integrand."

(12) Dr. DUNHAM JACKSON: "On approximation by trigonometric sums and polynomials."

(13) Dr. J. E. HODGSON: "Orthocentric properties of the plane directed n-line."

Mr. Williams was introduced by Professor Birkhoff. Dr. Hodgson's paper was communicated to the Society by Professor Morley. In the absence of the authors the papers of Professor Miller, Dr. Lefschetz, Dr. Urner, and Dr. Hodgson were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. By the use of the identity

$$2x(e^{2x} + e^{4x} + \cdots + e^{2nx}) = e^{(2n+1)x} \cdot \frac{2x}{e^x - e^{-x}} - \frac{2x}{1 - e^{-2x}}$$

Mr. Joffe expresses the sum of the odd, $(2k-1)$th (even,

$2k$th) powers of the first n positive integers as an integral function of $2n+1$ (even, of order $2k$; odd, of order $2k+1$). Noticing then that any even function of $2n+1$ is an integral function of n^2+n, he establishes the following theorem: The sum of the odd $(2k-1)$th powers of the natural numbers $1, 2, \cdots, n$ is an integral function (of order k) of n^2+n, with fractional coefficients. For the sum of even powers there is established a similar theorem, but in this case the integral function of n^2+n is multiplied by $2n+1$. The general form of the coefficients in both these cases is rather complicated; a few of the initial and terminal coefficients are simplified and are expressed in terms of Bernoullian numbers. Tables are appended giving the numerical values of the coefficients for all the functions corresponding to the first twenty-five powers.

2. Professor Schur recently called attention to several errors in a brief note by Professor Miller entitled "Groups generated by operators which transform each other into their inverses," [cf. *Jahrbuch über die Fortschritte der Mathematik*, volume 40 (1911), page 189]. In the present note Professor Miller develops the theory of these groups much more completely and observes that the true theorems relating to the points in question are even more general than the false ones which they aim to replace. Among these are the following: There is one and only one group of order 2^n, $n>2$, which can be generated by a set of m operators such that each of them transforms each of the $m-1$ remaining ones into its inverse and that no two of them are commutative. When $n=3$ this is the quaternion group, and when $n=4$ it is the Hamiltonian group of order 16, but it is not Hamiltonian for any larger value of n. It contains two and only two invariant operators when n is odd, and when n is even it involves exactly four such operators. These constitute the cyclic group of order 4 when n is of the form $4k+2$ and only then.

All the operators of this group of order 2^n have orders which divide 4, and all these operators of order 4 have a common square. This square generates the commutator subgroup. When n is of the form $4k+2$ exactly half of the operators of the group are of order 4, since the product of an invariant operator of order 4 and an operator of order 2 or 4 is of order 4 or 2 respectively in the group under consideration. The number of

the operators of order 2 for the other values of n can be readily stated in terms of general formulas.

3. Dr. Lefschetz's paper is an attempt to precise some notions relative to remarkable points of curves. Given a plane curve the arc of which is everywhere analytic, and a discrete aggregate of points and lines, curves of a given order m through λ of the points and tangent to μ of the lines will in general have a contact of order at most k with the curve at an arbitrary point. For certain points however the contact will be of order $k+1$, and these are defined as remarkable points. This definition is also extended to points of contact of multiply tangent curves, and to the similar points obtained by consideration of the reciprocal polar of the system with respect to any conic. It is then shown that, when m, λ, μ vary, the points defined belong to a discrete aggregate on the curve, and that the aggregate of lines of the plane that meet the curve in none of the remarkable points has the power of the continuum.

4. Dr. Urner discusses the behavior of a point transformation at a point where its Jacobian vanishes, attention being given chiefly to the transformation of contact of curves and surfaces. The notion of order of singularity is developed, and criteria are given for its determination.

5. A glance at the formulas for the integer transformation of the periods of the odd and even theta functions of p variables shows that the period and theta characteristics are transformed linearly and homogeneously under a collineation group in the finite domain S_{2p-1}, mod 2, which has an invariant null system. The transformations of the period characteristics contain the coefficients of the integer transformation linearly, i. e., they are transformed like the points (or their null S_{2p-2}'s) of the S_{2p-1}. The transformations on the theta characteristics contain the coefficients to the second degree but not homogeneously. Since $c^2 \equiv c$, mod 2, homogeneity can be effected and the theta characteristics are then transformed like a system of 2^{2p} quadrics in S_{2p-1}—precisely those quadrics whose polar systems coincide with the given null system. The even and odd characteristics correspond to quadrics of different type, containing respectively

$2^{p-1}(2^p+1)-1$ or $2^{p-1}(2^p-1)-1$ points. The object of Professor Coble's paper is to show that the entire theory of the dependence of these characteristics can be regarded as an image of the finite geometry and can be established with great facility from this point of view.

6. In a study of the finite quaternary linear groups with ordinary coefficients, some particular quaternary modular groups were noticed by Dr. Mitchell. A group of order 40,320 with the coefficients of the transformations in the $GF(3^2)$ was found to exist. This was shown to have a self-conjugate subgroup of order 20,160, holoedrically isomorphic with the ternary simple group consisting of all transformations with coefficients in the $GF(2^2)$ and determinant unity.

A group holoedrically isomorphic with the symmetric group on seven letters and with the coefficients of the transformations in the $GF(7)$ was also found to exist.

7. It is known that the invariants of the R^n are expressible in terms of the three-rowed determinants of the matrix of coefficients of the three n-ics in the parametric equations of the R^n. Dr. Rowe's paper consists of a formal statement of a method by means of which the undulation condition can be expressed as a determinant of order $4(n-3)$ and the cusp invariant as one of order $2(n-1)$, the constituents of these determinants being the three-rowed determinants mentioned above. An especially interesting feature is the relation which exists between the cusp and undulation conditions of the R^5 in the plane and the analogues of these two singularities of the R^{2k+1} in space of k dimensions.

8. Professor Osgood shows that a single-valued function of n complex variables, homogeneous and of dimension 0, which is analytic at every finite point of the space of these variables distinct from the point $(0, 0, \cdots)$, or has at most an unessential singularity, is a rational function of these variables.

9. Following investigations of Lebesgue on the subject of Fourier's series (*Bulletin de la Société Mathématique de France*, 1910), Dr. Jackson's paper is devoted to the study of the order of the approximation to a continuous function given by the partial sum, to terms of the nth degree of the expansion

of the function in series of Legendre's polynomials. It is found that, if $f(x)$ is a function such that the difference between two of its values does not exceed $\omega(\delta)$ when the difference of the corresponding values of the argument does not exceed δ, where $\omega(\delta)$ is any function which approaches zero with δ and satisfies one or two other restrictions, then the error of the approximation referred to, in the interior of the interval $(-1, 1)$, does not exceed in magnitude a quantity of the order of $\omega(1/n) \log n$. In particular, if $\omega(\delta) = \text{constant} \times \delta$ (Lipschitz condition), the upper limit for the error so obtained is of the order of $\log n/n$; and if $\lim_{\delta=0} \omega(\delta) \log \delta = 0$ (Lipschitz-Dini condition), this upper limit approaches zero when $n = \infty$, that is, the series converges. An analogous result is obtained when it is assumed that $f(x)$ has a $(k-1)$th derivative satisfying a Lipschitz condition, the corresponding upper limit being of the order of $\log n/n^k$.

It is further demonstrated by an example that the hypothesis made in the first theorem is not sufficient in itself to ensure any more rapid rate of convergence than that which was actually shown to be attained, so that the theorem may to this extent be regarded as satisfactory.

10. In this paper Mr. Williams considers the system of non-homogeneous linear difference equations

$$g_i(x+1) = \sum_{j=1}^{n} a_{ij}(x)\, g_j(x) + b_i(x) \qquad (i = 1, 2, \cdots, n),$$

where the functions $a_{ij}(x)$ and $b_i(x)$ are such that

$$a_{ij}(x) = x^{\mu}\left(a_{ij} + \frac{a_{ij}^{(1)}}{x} + \frac{a_{ij}^{(2)}}{x^2} + \cdots\right),$$

$$b_i(x) = x^{\nu}\left(b_i + \frac{b_i^{(1)}}{x} + \frac{b_i^{(2)}}{x^2} + \cdots\right)$$

$$(i, j = 1, 2, \cdots, n;\ |x| > R).$$

By a direct use of the well known sum formula, obtained by the method of variation of "constants," which has previously been regarded as purely formal, it is shown that in general there exist two simple solutions $g_{11}(x)$, $g_{21}(x)$, $\cdots$, $g_{n1}(x)$ and $g_{12}(x)$, $g_{22}(x)$, $\cdots$, $g_{n2}(x)$.

The first solution is analytic throughout the finite plane,

save for possible singularities at the singularities of the functions $a_{ij}(x)$, $b_i(x)$, and the zeros of the determinant $|a_{ij}(x)|$, and points any number of units to the left of these points; the second solution is analytic except at points any number of units to the right of the singularities of $a_{ij}(x)$, $b_i(x)$. These two solutions are shown to be asymptotic, in the right and left half planes respectively, to the series formally satisfying the given system; and one of them maintains its asymptotic form in the remaining half plane at a sufficient distance from the axis of reals.

The method employed to make the sum formula yield these solutions is to replace the sums in one case by an infinite series, and in the other by an appropriate contour integral.

Like results are derived for a single non-homogeneous linear difference equation of the nth order, reducible to such a system.

11. In dealing with the isoperimetric problem of the calculus of variations, where one is asked to find and discuss the properties of curves

$$D: \quad x = \varphi(t), \quad y = \psi(t)$$

which minimize the integral

$$J \equiv \int F(x, y, x', y')dt$$

subject to the condition

$$K \equiv \int G(x, y, x', y')dt = L,$$

it is customary to assume that the functions F and G are continuous in their four arguments. In Dr. Miles's note some properties of the minimizing curve D are given when the functions F and G are allowed to have a finite discontinuity as the point (x, y) passes through a given curve M.

12. In his thesis (Göttingen, 1911) Dr. Jackson has proved the following theorem: If $f(x)$ is a function which possesses a derivative of order $k-1$ satisfying a Lipschitz condition throughout a closed interval, then $f(x)$ may be approximately represented in the interval by a polynomial of degree n or lower, with a maximum error not exceeding K/n^k, where k is independent of n. The most interesting special case is that in which $k = 1$ and $f(x)$ itself satisfies a Lipschitz condition. On the assumption that $f(x)$ has the period 2π and

satisfies the conditions stated above for all values of x, it was shown that a precisely analogous theorem holds for the approximation of $f(x)$ by a trigonometric sum of order n or lower, this result being obtainable as a consequence of the preceding. It is now shown that decided simplification in the proof of both theorems may be effected by proving the second directly (this had been done only for $k = 1$) and deducing the first from it.

This method has the further advantage that the numerical constants involved can be computed more conveniently. For example, if $f(x)$ satisfies the condition

$$|f(x_2) - f(x_1)| \leqq |x_2 - x_1|$$

in the closed interval (0, 1), it can be approximately represented in this interval by a polynomial of degree n or lower, with an error which never exceeds $3/n$, for all positive integral values of n. The same line of investigation leads to results in the theory of Fourier's series.

13. There is a theorem that the perpendiculars let fall from the incenters of three out of four lines of given direction upon the remaining line touch a circle. In Dr. Hodgson's paper a circle is obtained for any even number of lines, beginning with four. If we take this circle for any $2n$ out of $2n + 1$ lines, the $2n + 1$ circles touch a line. The question of the reversal of direction of one or more of $2n$ lines is then taken up, and this is followed by the consideration of the configuration of circles arising from four, five, and six lines.

F. N. COLE,
Secretary.

ON THE FOUNDATIONS OF THE THEORY OF LINEAR INTEGRAL EQUATIONS.*

BY PROFESSOR E. H. MOORE.

1. *The Analogous Systems of Linear Equations.*

THE theory of linear integral equations, mathematically considered, has its taproot in the classical analogies between

*Address of the Vice-President and Chairman of Section A of the American Association for the Advancement of Science, Washington, December 29, 1911.

an algebraic sum, the sum of an infinite series, and a definite integral.

Consider the linear algebraic equation

$$(\mathrm{I}^\circ) \qquad x = ky$$

for the number y, the coefficient k and the number x being given. From this single equation I° we ascend to the algebraic system

$$(\mathrm{II}_n^\circ) \qquad x_i = \sum_{j=1}^{n} k_{ij} y_j \qquad (i = 1, 2, \cdots, n)$$

of n simultaneous linear equations for the determination of the set (y_i) of n numbers $y_1, \cdots, y_n$, the matrix (k_{ij}) of n^2 coefficients $k_{11}, \cdots, k_{nn}$ and the set (x_i) of n numbers $x_1, \cdots, x_n$ being given.

To this algebraic system (II_n°) we have by the classic analogy the two corresponding transcendental systems

$$(\mathrm{III}^\circ) \qquad x_i = \sum_{j=1}^{\infty} k_{ij} y_j \qquad (i = 1, 2, \cdots);$$

$$(\mathrm{IV}^\circ) \qquad \xi(s) = \int_a^b \kappa(s, t)\eta(t)dt \qquad (a \leqq s \leqq b).$$

In III° the infinite set (y_i) is to be found, the infinite set (x_i) and the infinite matrix (k_{ij}) of coefficients being given; the suffixes i, j have the range $1, 2, \cdots$. In IV° the unknown function η and the known function ξ are functions of one variable ranging over the interval a-b of the real number system, while the known coefficient function or, in Hilbert's terminology, kernel κ is a function of two variables ranging independently over that interval. It is plain that the theories of III°, IV° must involve convergence considerations. Throughout, the numbers and the functional values of the functions are real or complex numbers.

You are aware that the study of the algebraic system II_n°, initiated before 1678 by the genial intuition of the philosopher-mathematician Leibniz, led to the development of the theory of determinants—a theory which in the nineteenth century came to permeate all branches of number theory, algebra, analytic geometry, and pure and applied analysis, exerting everywhere a profound influence not merely by its usefulness but perhaps even more by the extreme elegance of its methods and results.

The theory of infinite determinants connected with the denumerably infinite system III°, or more exactly with the equivalent system in which the indices i, j have the integral values from $-\infty$ to $+\infty$, was initiated by G. W. Hill, who in 1877 made hardy but happily effective use of the determinant of a system of the latter type in his solution of a differential equation arising in his memorable study of the motion of the lunar perigee. To supply the requisite convergence proofs, Henri Poincaré in 1886 laid the foundations of the general theory of infinite determinants, which has since been developed chiefly by Helge von Koch.

We are led to an analogous determinant, not from the continuously infinite system IV° of linear equations, but from the system

$$\text{(IV)} \qquad \xi(s) = \eta(s) - z\int_a^b \kappa(s, t)\eta(t)dt \quad (a \leqq s \leqq b).$$

Here z is a given number, real or complex, and we consider the *regular* case, in which the functions involved are continuous real or complex valued functions of their arguments.

The corresponding systems

$$\text{(I)} \qquad x = y - zky,$$

$$\text{(II}_n\text{)} \qquad x_i = y_i - z\sum_{j=1}^{n} k_{ij}y_j \quad (i = 1, 2, \cdots, n),$$

$$\text{(III)} \qquad x_i = y_i - z\sum_{j=1}^{\infty} k_{ij}y_j \quad (i = 1, 2, \cdots)$$

are respectively equivalent to the systems I°, II$_n$°, III°.

The types IV, IV° are however essentially distinct. For instance, if we look at IV and IV° as transformations of the functions η into the functions ξ, the type IV contains (in the case of vanishing parameter z or identically vanishing coefficient function or kernel κ) the identical transformation $\xi = \eta$, while the type IV° does not contain this transformation.

The solution η of the system or integral equation IV may be expanded formally as a power series in the parameter z, viz.,

$$\eta(p) = \xi(p) + z\int_a^b \kappa(p, p_1)\xi(p_1)dp_1 + z^2\int_a^b\int_a^b \kappa(p, p_1)\kappa(p_1, p_2)\xi(p_2)dp_1dp_2 + \cdots.$$

This power series converges near $z = 0$ uniformly in p, and accordingly for z sufficiently small has as sum a continuous function η which is readily proved to be a solution of IV.

Impelled by the fact that integral equations of type IV occur very frequently in the linear problems of mathematical physics, in the late nineties of the last century the Swedish mathematical physicist Ivar Fredholm undertook the study of the analytic character of the solution η as a function of the parameter z. After earlier notes on the subject, Fredholm published his fundamental memoir in 1903 in volume 27 of the *Acta Mathematica*, in one of the two volumes of the *Acta* dedicated to the memory of Abel on the occasion of the centenary of his birth. And this was the more fitting since Abel first studied special integral equations of the type IV°. Accordingly Fredholm calls the equation IV° Abel's integral equation. Mathematicians generally call the equation IV Fredholm's integral equation. With Hilbert one also designates the equations IV°, IV as integral equations of the first and second kind.

Fredholm found that the function η is a single-valued analytic function of the parameter z, having at most polar singularities in the finite z-plane, and he exhibited it explicitly as the quotient of two permanently converging power series in z. The denominator series with coefficients depending only on the kernel κ, viz.,

$$1-z\int_a^b \kappa(p, p)dp+\frac{z^2}{2}\int_a^b\int_a^b \begin{vmatrix} \kappa(p_1, p_1) & \kappa(p_1, p_2) \\ \kappa(p_2, p_1) & \kappa(p_2, p_2) \end{vmatrix} dp_1dp_2 \pm \cdots,$$

is Fredholm's determinant of the kernel κ with parameter z. In case z is not a root of this determinant, for every function ξ there is a definite solution η of the equation IV, and the same is true as to the adjoint equation

$$(\breve{\text{IV}}) \qquad \xi(t) = \eta(t) - z\int_a^b \eta(s)\kappa(s, t)ds \quad (a \leqq t \leqq b).$$

On the other hand, if z is a root of the determinant, it is of finite multiplicity m, and each of the corresponding homogeneous equations

$$(\text{IV}_H) \qquad \eta(s) = z\int_a^b \kappa(s, t)\eta(t)dt \qquad (a \leqq s \leqq b),$$

$$(\breve{\text{IV}}_H) \qquad \eta(t) = z\int_a^b \eta(s)\kappa(s, t)ds \qquad (a \leqq t \leqq b)$$

has a solution η not identically vanishing, the number n of linearly independent solutions η for one equation being the same as for the other equation and at most m.

These few results suffice to suggest the close parallelism between Fredholm's theory of the integral equation IV and the current theory of the algebraic system II_n.

We have seen, then, that the theories of determinants of the matrices or kernels of the three types—the finite, the denumerably infinite, the continuously infinite—were initiated by the mathematician-philosopher Leibniz, the mathematical astronomer Hill, the mathematical physicist Fredholm; and we appreciate anew the magnitude of the debt owed by pure mathematics to its most closely related sister sciences—logic, astronomy, physics—a debt abundantly repaid by the applications throughout the wide range of the sciences, at least in the progress of time, of even the most abstract doctrines of pure mathematics.

2. *References to the Literature.*

The investigations of von Koch and Fredholm opened the way for the systematic development, now in rapid progress, of the analogies and the interrelations between the algebraic and the two kinds of transcendental theories, and for the immediate application of the new results in various domains of pure and applied analysis.

Especially noteworthy are the memoirs of David Hilbert and of Erhard Schmidt. By direct limiting processes Hilbert obtains from algebraic theorems the corresponding transcendental theorems. Hilbert has thus initiated a theory of functions of a denumerable infinity of variables, from which in turn, by the connection between functions of continuous variables and their Fourier coefficients, one proceeds to the theory of functions of continuous variables. In particular, Hilbert has studied real-valued symmetric kernels, obtaining the transcendental analogues, for example, of the orthogonal transformation of the algebraic quadratic form to the sum of squares of linear forms. Geometric analogies of metrical nature play a considerable rôle in the work of Hilbert, and perhaps even more in the work of Schmidt, who treats transcendental problems directly, using methods originated by H. A. Schwarz in the potential theory. Schmidt has also entered upon the study of non-linear integral equations.

But for details of the extensive literature and present state of the whole subject I must content myself with referring to the most recent books:

HEYWOOD ET FRÉCHET: L'équation de Fredholm et ses applications à la physique mathématique. Hermann, Paris, 1912.

LALESCO: Introduction à la théorie des équations intégrales. Hermann, Paris, 1912.

and to the reports:

HILBERT: "Wesen und Ziele einer Analysis der unendlichvielen unabhängigen Variabeln." *Palermo Rendiconti*, volume 27, pages 59–74 (1909).

VON KOCH: "Sur les systèmes d'une infinité d'équations linéaires à une infinité d'inconnues." *Compte rendu du Congrès des Mathématiciens*, tenu à Stockholm 22–25 Septembre, 1909, pages 43–61; Teubner, Leipzig, 1910.

FREDHOLM: "Les équations intégrales linéaires." Ibid., pages 92–100.

BATEMAN: "Report on the history and present state of the theory of integral equations." Report to the British Association for the Advancement of Science, Sheffield, 1910, 80 pp.; Burlington House, London, 1911.

HAHN: "Bericht über die Theorie der linearen Integralgleichungen." *Jahresbericht der Deutschen Mathematiker-Vereinigung*, volume 20, pages 69–117 (1911).

3. *The Fundamental Problem of Unification. General Analysis.*

We are now in position to take up, as the specific subject of this discourse, the question of foundations of the theory of linear integral equations. We have seen that the algebraic theory serves to suggest the corresponding transcendental theories, or even to determine those theories by suitable use of limiting processes. But this state of affairs may be recognized as only preliminary to the determination of a general theory capable of specialization into the various theories. This is in accordance with a general heuristic principle of scientific procedure, which I have formulated as follows:

The existence of analogies between central features of various theories implies the existence of a general theory which underlies the particular theories and unifies them with respect to those central features.

For the case of the real-valued symmetric kernel, and in fact for the more general case of the complex-valued hermitian kernel, i. e., a kernel κ satisfying identically the condition that $\kappa(s, t)$ and $\kappa(t, s)$ are conjugate complex numbers, I

took up six years ago this problem of unification for the Hilbert theory as presented by Schmidt. This was the theme of my series of lectures: "On the theory of bilinear functional operations," at the colloquium of the American Mathematical Society, held in September, 1906, in New Haven, under the auspices of Yale University.

Subsequent study led to the recognition that the general theory of linear integral equations is merely a division in the theory of a certain form of general analysis, an introduction to which, instead of the colloquium lectures, I published,* as pages 1–150, in the volume The New Haven Mathematical Colloquium, etc., Yale University Press, New Haven, 1910.

This morning I wish to establish, in the sense of general analysis, an adequate and satisfactorily simple foundation for the general theory of linear integral equations, embracing by specialization, as we shall see, together with an interesting variety of new theories, the algebraic and both types of transcendental theories, now current, at least in so far as regular kernels are concerned.

As to the system III, I call the matrix-kernel (k_{ij}) regular in case there is a set (k_i) of numbers of finite norm $\Sigma_i |k_i|^2$, such that for every i and j $|k_{ij}| \leqq |k_i k_j|$. This regular kernel satisfies the latest condition found † by von Koch as sufficient that the infinite determinant and all its minors converge absolutely. Then, if the sets (x_i), (y_i) are of finite norm, the system III may be treated either by the method of infinite determinants or by Hilbert's theory of functions of infinitely many variables.

I shall indicate, first, the terminology or *basis* of the foundation of the general theory, and then two sets of *postulates*, the former effective for the validation of the general Fredholm theory for the general kernel, and the latter effective for the validation of the general Hilbert-Schmidt theory of the real symmetric or the more general hermitian kernel. And, in advance, I notice that for the latter theory we need more postulates than for the former theory. This is in accordance

* Cf. also my paper, "On a form of General Analysis, with application to linear differential and integral equations," read before the Section on Analysis of the Rome Congress of 1908, Atti, etc., vol. 2 (1909), pp. 98–114.

† Loc. cit., pp. 49, 50; *Palermo Rendiconti*, vol. 28 (1909), pp. 257, 263. The condition is that the kernel (k_{ij}) have the form $k_{ij} = u_{ij} v_i / v_j \quad (i, j)$, where $v_i \neq 0 \quad (i)$ and the series $\Sigma_{ij} u_{ij}^2$, $\Sigma_i u_{ii}$ converge absolutely.

with the nature of things logical: *we must pay for the elaboration of theory by the imposition of additional postulates and the corresponding restriction of scope of application.*

4. *Fredholm's Equation in General Analysis. The Basis* Σ_1.

In order to bring the equation-systems I, II, III into notational conformity with the integral equation IV, we regard, for instance, in III the set (x_i) of numbers x_i $(i = 1, 2, 3, \cdots)$ as a function x or ξ of the argument i or s with the range $i = s = 1, 2, 3, \cdots$. Then I, II_n, III, IV are special cases of the general equation

$$(G) \qquad \xi = \eta - zJ\kappa\eta,$$

with the meaning

$$(G) \qquad \xi(s) = \eta(s) - zJ_t\kappa(st)\eta(t) \quad (s),$$

which we designate as Fredholm's equation in general analysis. The kernel κ, the parameter z, and the function ξ being given, the function η is to be determined as a solution of the equation G.

The understanding is that (1) ξ and η are functions of an argument p or s or t having a certain range $\mathfrak{P}$; (2) κ or $\kappa(st)$ is a function of two arguments ranging independently over $\mathfrak{P}$; (3) J or J_t is a functional operation turning a product $\kappa\eta$ or $\kappa(st)\eta(t)$ into a function of the argument s; and (4) the equation G holds for every value of s on the range $\mathfrak{P}$.

For the general theory this range $\mathfrak{P}$ is simply a class of elements p. These elements p are of any nature whatever, e. g., numbers, sets of numbers, functions, points, curves; and they are not necessarily all of the same nature. Thus, the range $\mathfrak{P}$ is a general class of general elements. This "general" is the *true general*, in the sense of *arbitrarily special*, that is, capable of arbitrary specification—*without the exclusion* of exceptional or singular cases.

Thus, for the general theory of the equation G the range $\mathfrak{P}$ enters without the imposition of restrictive properties or features, and it is *this presence in the theory of a general class* which *constitutes the theory a doctrine of the form of general analysis* which we are developing.

For the respective instances II_n, III, IV, the range $\mathfrak{P}$ is

finite, denumerably infinite, continuously infinite, consisting of the respective elements

$$p = 1, 2, \cdots, n; \quad p = 1, 2, 3, \cdots; \quad a \leqq p \leqq b.$$

We denote these ranges by the notations

$$\mathfrak{P}^{\mathrm{II}_n}; \quad \mathfrak{P}^{\mathrm{III}}; \quad \mathfrak{P}^{\mathrm{IV}}$$

The respective functional operations J are

$$J_t^{\mathrm{II}_n} \equiv \sum_{t=1}^{n}; \quad J_t^{\mathrm{III}} \equiv \sum_{t=1}^{\infty}; \quad J_t^{\mathrm{IV}} \equiv \int_a^b dt\,.$$

In the instances III and IV the functions ξ, η, κ are necessarily subject to certain conditions of convergence or of continuity. The conditioning properties are defined in terms of features possessed by the special classes $\mathfrak{P}^{\mathrm{III}}$, $\mathfrak{P}^{\mathrm{IV}}$; we are able to speak in $\mathfrak{P}^{\mathrm{III}}$ of p tending to ∞, and in $\mathfrak{P}^{\mathrm{IV}}$ of the difference $p_1 - p_2$ of two elements p. Similarly, in the general theory the functions must possess certain properties, which must however be postulated and not explicitly defined, since we attribute to the general range no features available for use in the definitions. Now, instead of postulating *properties of the functions,* it is technically more convenient to postulate *classes of functions* to which the functions shall belong, viz., the classes of functions are the classes of all functions possessing the respective properties.

Accordingly, the form of the general equation G leads to the following first basis:

$$\Sigma_1 = (\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M};\ \mathfrak{K};\ J)$$

for the construction of a general theory of the linear equation G. Here $\mathfrak{A}$ denotes the class of all real or the class of all complex numbers a; $\mathfrak{P}$ denotes a general class of general elements p or s or t or u or v or w; $\mathfrak{M}$ denotes a class of single-valued functions μ on $\mathfrak{P}$ to $\mathfrak{A}$, that is, for every function μ and argument p, $\mu(p)$ denotes a definite number a of the class $\mathfrak{A}$; $\mathfrak{K}$ denotes a class of functions κ on $\mathfrak{P}\mathfrak{P}$ to $\mathfrak{A}$, that is, for every function κ and (ordered) pair (st) of arguments of $\mathfrak{P}$, $\kappa(st)$ denotes a definite number a of the class $\mathfrak{A}$; and J or J_t denotes a definite functional operation on $\mathfrak{K}\mathfrak{M}$ or $\mathfrak{K}_{st}\mathfrak{M}_t$ to $\mathfrak{M}$ or $\mathfrak{M}_s$, that is, for every function κ of $\mathfrak{K}$ and η of $\mathfrak{M}$, $J\kappa\eta$ or $J_t\,\kappa(st)\eta(t)$ denotes a definite function of $\mathfrak{M}$ or $\mathfrak{M}_s$.

For the basis Σ_1 *the problem of foundations* of a general theory of the equation G is then: *to specify properties of the constituents of the basis* Σ_1 *sufficient to validate the desired theory.* The range $\mathfrak{P}$ is to remain general, and the properties specified are to be *of general reference,* that is, defined with respect to the general range $\mathfrak{P}$.

It is convenient here to refer to the important memoirs of S. Pincherle:

"Sulle equazioni funzionali lineari". *Rendiconti della R. Accademia dei Lincei,* ser. 5, vol. 14 (1905), pp. 366–374;
"Sulle equazioni funzionali lineari". *Bologna Memorie,* ser. 6, vol.3 (1906), pp. 143–171;
"Appunti di calcolo funzionali". *Bologna Memorie,* ser. 6, vol. 8 (1911), pp. 1–38.

In these memoirs, from the standpoint of the general theory of linear distributive functional operations, Pincherle investigates the problem of foundations for a theory of the equation

$$(G') \qquad \xi = \eta - z J'\eta,$$

which includes the equation G, with the basis

$$\Sigma' = (\mathfrak{A}; \mathfrak{P}; \mathfrak{M}; J'),$$

where J' is a functional operation on $\mathfrak{M}$ to $\mathfrak{M}$.

5. *Certain Definitions. The Closure Property* C_1. *Relative Uniformity of Convergence.*

We do not however retain the basis Σ_1. With the purpose of obtaining finally a general theory characterized by its simplicity and by its possession of a number of important *closure properties,* we set up other bases $\Sigma_2, \cdots, \Sigma_6$. Whenever a general theory, as a matter of fact, includes, as a special instance, a theory analogous to but not *a priori* one of its instances, we speak of a closure property of the general theory. The terminology is adapted from that in use in the theory of point sets.

That the general theory include, as special instances, the current theories of the equations I, II_n, III, IV with regular kernels is the closure property C_1, fundamental to the whole inquiry.

You are familiar with the notion, *uniformity of convergence* of a sequence of functions over a range of values of the argument of the functions, and you appreciate the fundamental

rôle played by the notion in the development of analysis during the last half century. A sequence $\{\mu_n\}$ of functions μ_n $(n = 1, 2, 3, \cdots)$ of the argument p on the range $\mathfrak{P}$ converges over the range $\mathfrak{P}$ uniformly to a function θ as limit function, in notation

$$\underset{n}{L}\,\mu_n = \theta \quad (\mathfrak{P}),$$

in case for every positive number e and index n greater than a number n_e, dependent upon e alone, the difference $\mu_n(p) - \theta(p)$ is, for every value of p on the range $\mathfrak{P}$, in absolute value at most e:

$$|\,\mu_n(p) - \theta(p)\,| \leqq e.$$

For investigations in general analysis we need a more general notion, the notion of *relative uniformity of convergence*, or, *uniformity of convergence relative to a scale function*. If the function σ (defined on the same range $\mathfrak{P}$) is the scale function, this relative uniformity, in notation

$$\underset{n}{L}\,\mu_n = \theta \quad (\mathfrak{P};\ \sigma),$$

has the same definition except that the final inequality is replaced by

$$|\,\mu_n(p) - \theta(p)\,| \leqq e\ |\sigma(p)|\,.$$

We speak also of relative uniformity as to a class $\mathfrak{S}$ of scale functions; the notation

$$\underset{n}{L}\,\mu_n = \theta \quad (\mathfrak{P};\ \mathfrak{S})$$

means that for some function σ of the class $\mathfrak{S}$ we have the relative uniformity as to the scale function σ.

One observes that the classical uniformity is that instance of relative uniformity in which the scale function is identically 1.

In illustration of this notion of relative uniformity, consider a sequence $\{\mu_n\}$ of real-valued nowhere negative functions μ_n of the real variable p on the infinite interval $\mathfrak{P} = 1 - \infty$ of the number system, and let the functions μ_n be individually integrable from 1 to ∞. If relative to a scale function σ of the same kind the sequence $\{\mu_n\}$ converges uniformly on $\mathfrak{P}$ to a limit function θ, then θ also is integrable from 1 to ∞, and the integral of θ is the limit as to n of the integral of μ_n. But the limit function θ is no longer necessarily integrable

from 1 to ∞, if the convergence is merely uniform in the classical sense ($\sigma = 1$), as appears from the following example:

$$\cdots\cdots\cdots\cdots$$
$$\mu_n(p) = 1/p \quad (1 \leqq p \leqq n), \quad n/p^2 \ (p \geqq n),$$
$$\theta(p) = 1/p \qquad\qquad (1 \leqq p).$$

To facilitate the exposition of the sequel we need certain additional definitions.

Consider a class $\mathfrak{M}$ of functions μ on $\mathfrak{P}$ to $\mathfrak{A}$, that is, on the range $\mathfrak{P}$ with functional values belonging to the class $\mathfrak{A}$. The class $\mathfrak{M}_L$, *the linear extension of the class* $\mathfrak{M}$, is the class of all functions μ_L of the form

$$\mu_L = a_1\mu_1 + a_2\mu_2 + \cdots + a_n\mu_n,$$

viz., the class of all linear combinations of a finite number of functions belonging to the class $\mathfrak{M}$ with numerical coefficients belonging to the class $\mathfrak{A}$. Further, $\mathfrak{S}$ being a class of functions σ on $\mathfrak{P}$ to $\mathfrak{A}$, the class $\mathfrak{M}_{\mathfrak{S}}$, *the class* $\mathfrak{M}$ *extended as to the class* $\mathfrak{S}$, is the class of all functions θ of the form

$$\theta = \underset{n}{L}\,\mu_n \quad (\mathfrak{P};\ \sigma),$$

a form which has been defined above. The class $\mathfrak{M}$ is contained in the class $\mathfrak{M}_L$ and for every $\mathfrak{S}$ in the class $\mathfrak{M}_{\mathfrak{S}}$. If the classes $\mathfrak{M}$ and $\mathfrak{M}_L$ are identical, the class $\mathfrak{M}$ is said to be *linear*, to have the property L. If the classes $\mathfrak{M}$ and $\mathfrak{M}_{\mathfrak{M}}$ are identical, the class $\mathfrak{M}$ is said to be *closed*, to have the property C; otherwise expressed, the class $\mathfrak{M}$ is closed in case every function θ of the form

$$\theta = \underset{n}{L}\,\mu_n \quad (\mathfrak{P};\ \mu)$$

belongs to the class $\mathfrak{M}$. The class $\mathfrak{M}_*$, *the* $*$*-extension of the class* $\mathfrak{M}$, is the class $(\mathfrak{M}_L)_{\mathfrak{M}}$, that is, the extension as to $\mathfrak{M}$ of the linear extension of $\mathfrak{M}$. The class $\mathfrak{M}^2$ is the class of all functions of the form $\mu_1\mu_2$ or $\mu_1(p)\mu_2(p)$, that is, of all products of pairs of functions of the class $\mathfrak{M}$, the arguments of the two functions being the same.

In illustration of these definitions, if $\mathfrak{M}$ is the class $\mathfrak{M}^{\mathrm{IV}}$ of all continuous functions on the finite linear interval $\mathfrak{P}^{\mathrm{IV}}$, a-b, we have

$$\mathfrak{M} = \mathfrak{M}_L = \mathfrak{M}_{\mathfrak{M}} = \mathfrak{M}_* = \mathfrak{M}^2.$$

Further, if $\mathfrak{M}$ is the class $\mathfrak{M}^{\mathrm{III}_2}$ of all functions μ on $\mathfrak{P}^{\mathrm{III}}$

$(p = 1, 2, 3, \cdots)$ such that the series $\Sigma_p \mu(p)^2$ converges absolutely, we have

$$\mathfrak{M} = \mathfrak{M}_L = \mathfrak{M}_{\mathfrak{M}} = \mathfrak{M}_*; \quad \mathfrak{M}^2 = \mathfrak{M}^{\mathrm{III}_1},$$

where $\mathfrak{M}^{\mathrm{III}_1}$ denotes the class of all functions μ on $\mathfrak{P}$ such that the series $\Sigma_p \mu(p)$ converges absolutely. Thus the classes $\mathfrak{M}^{\mathrm{III}_2}$ and $\mathfrak{M}^{\mathrm{IV}}$ occurring in the regular cases of equations III and IV are linear closed classes of functions.

Consider further two general ranges $\mathfrak{P}'$, $\mathfrak{P}''$ conceptually but not necessarily actually distinct. The product range $\mathfrak{P}'\mathfrak{P}''$ is the class of all composite elements (p', p'') or $p'p''$, the first constituent p' being an element p' of the class $\mathfrak{P}'$, and the second constituent p'' being an element p'' of the class $\mathfrak{P}''$. For example, if $\mathfrak{P}'$ is the linear interval a'-b' and $\mathfrak{P}''$ is the linear interval a''-b'', the product $\mathfrak{P}'\mathfrak{P}''$ is, to speak geometrically, the rectangle $a' \leqq p' \leqq b'$, $a'' \leqq p'' \leqq b''$. The *product class* $\mathfrak{M}'\mathfrak{M}''$ of two classes $\mathfrak{M}'$, $\mathfrak{M}''$ of functions on the respective ranges $\mathfrak{P}'$, $\mathfrak{P}''$ consists of all products $\mu'\mu''$ or $\mu'(p')\mu''(p'')$ of a function μ' of the class $\mathfrak{M}'$ on $\mathfrak{P}'$ and a function μ'' of the class $\mathfrak{M}''$ on $\mathfrak{P}''$. The class $(\mathfrak{M}'\mathfrak{M}'')_*$, *the* $*$*-composite of two classes* $\mathfrak{M}'$, $\mathfrak{M}''$ on the respective ranges $\mathfrak{P}'$, $\mathfrak{P}''$ is, as indicated by the notation, the $*$-extension of the product class $\mathfrak{M}'\mathfrak{M}''$, viz., the extension as to the product class $\mathfrak{M}'\mathfrak{M}''$ of the linear extension of the product class $\mathfrak{M}'\mathfrak{M}''$. The classes $\mathfrak{M}'\mathfrak{M}''$ and $(\mathfrak{M}'\mathfrak{M}'')_*$ are classes of functions on the product range $\mathfrak{P}'\mathfrak{P}''$, and accordingly, if the ranges $\mathfrak{P}'$, $\mathfrak{P}''$ are identical, $\mathfrak{P}' = \mathfrak{P}'' \equiv \mathfrak{P}$, the arguments of the functions of those classes are variables $(p', p''$ or $p_1, p_2)$ ranging independently over $\mathfrak{P}$.

The suitability of these notions for use in a general theory of linear integral equations is indicated by the fact that for the regular case of the equation III or IV the functions ξ and η belong to a class $\mathfrak{M}$ ($\mathfrak{M}^{\mathrm{III}_2}$ or $\mathfrak{M}^{\mathrm{IV}}$) whose $*$-composite with itself is the class $\mathfrak{K}$ to which the kernels belong, viz.,

$$\mathfrak{K} = (\mathfrak{M}\mathfrak{M})_*.$$

6. *The Bases* Σ_2, Σ_3, Σ_4.

With the aid of the notions and notations now at hand we are able to proceed rapidly towards our goal. We recall that the basis Σ_1

$$\Sigma_1 = (\mathfrak{A}; \mathfrak{P}; \mathfrak{M}; \mathfrak{K}; J),$$

was dictated by the mere form of the general equation G

$$(G) \qquad \xi(s) = \eta(s) - zJ_t\,\kappa(st)\eta(t) \quad (s).$$

By the consideration that in the typical instances I, II_n, III, IV the kernel $\kappa(st)$ for every s as a function of the argument t belongs to the class $\mathfrak{M}_t$ to which the function $\eta(t)$ belongs, we are led to a basis Σ_2, simpler than Σ_1, in the form

$$\Sigma_2 = (\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M};\ \mathfrak{N} \equiv \mathfrak{M}^2;\ \mathfrak{K};\ J).$$

Here the class $\mathfrak{N}$ is defined as the class $\mathfrak{M}^2$ of all products $\mu_1\mu_2$ of pairs of functions of the class $\mathfrak{M}$, the arguments being the same for the two functions; and J is a functional operation on $\mathfrak{N}$ to $\mathfrak{A}$, so that for every function ν of $\mathfrak{N}$, $J\nu$ denotes a number a of the class $\mathfrak{A}$.

This system Σ_2 was basal for my lectures at the New Haven Colloquium of 1906. The development of the theory of the general equation G on the basis Σ_2 requires numerous postulates. We must, for instance, arrange to extend the scope of the functional operation J from the class $\mathfrak{N}$ to its linear and $*$-extensions, $\mathfrak{N}_L$ and $\mathfrak{N}_*$, in such wise that from the equations

$$\nu_L = a_1\nu_1 + \cdots + a_n\nu_n; \quad \theta = \underset{n}{L}\nu_{Ln} \quad (\mathfrak{P};\ \nu)$$

we have the conclusions

$$J\nu_L = a_1J\nu_1 + \cdots + a_nJ\nu_n; \quad J\theta = \underset{n}{L}J\nu_{Ln}.$$

Accordingly, if we define the class $\mathfrak{N}$ as the class $\mathfrak{M}_*$, the $*$-extension of the class $\mathfrak{M}$, and take J as a functional operation on this class $\mathfrak{N}$ to $\mathfrak{A}$, we have a simpler basis Σ_3

$$\Sigma_3 = (\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M};\ \mathfrak{N} \equiv \mathfrak{M}_*;\ \mathfrak{K};\ J).$$

A theory based on Σ_3 requires postulates involving the kernel class $\mathfrak{K}$. These postulates are avoided if we define the class $\mathfrak{K}$ as the $*$-composite of the class $\mathfrak{M}$ with itself. Thus we obtain the still simpler basis Σ_4

$$\Sigma_4 = (\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M};\ \mathfrak{N} \equiv \mathfrak{M}_*;\ \mathfrak{K} \equiv (\mathfrak{M}\mathfrak{M})_*;\ J).$$

Here J is still a functional operation on $\mathfrak{N}$ to $\mathfrak{A}$. For the regular cases of the typical instances I, II_n, III, IV the definition just suggested of the kernel class $\mathfrak{K}$ is appropriate.

As to irregular cases, it is clear that a greater variety could be treated on the basis Σ_2 or Σ_3 than on the basis Σ_4.

7. *The Closure Property C_2. The Basis Σ_5.*

In response to the desire that our theory of the general equation G

$$(G) \qquad \xi(s) = \eta(s) - zJ_t\,\kappa(st)\eta(t) \quad (s)$$

shall possess a certain closure property C_2, in addition to the fundamental property C_1 of having as special instances the current theories for the regular cases of the equations (I, II_n, III, IV), we are led to the basis Σ_5.

Under the postulates to be imposed on the class $\mathfrak{M}$, the kernel $\kappa(st)$ of the class $\mathfrak{K} \equiv (\mathfrak{M}\mathfrak{M})_*$ is, as in the typical instances, for every s as a function of t of the class $\mathfrak{M}_t$ to which also $\eta(t)$ belongs: thus, the operation J enters the equation G in the form

$$J\alpha\beta \quad \text{or} \quad J_p\alpha(p)\beta(p),$$

where α and β are functions of the class $\mathfrak{M}$. Now for the instance II_n the expression

$$J^{\mathrm{II}_n}\alpha\beta \quad \text{or} \quad \sum_{p=1}^{n}\alpha(p)\beta(p)$$

is Grassmann's inner product of the n-dimensional vectors

$$\alpha = (\alpha(1), \cdots, \alpha(n)); \quad \beta = (\beta(1), \cdots, \beta(n)),$$

and accordingly we term the expression $J_p\alpha(p)\beta(p)$ or $J\alpha\beta$ the *inner product* of the functions α, β. In this we follow usage already established for the instances III, IV.

For the basis Σ_4 it is convenient temporarily to denote the operation J by J^4 and the equation G by G^4. J^4 is then a functional operation on $\mathfrak{N} \equiv \mathfrak{M}_*{}^2$ to $\mathfrak{A}$. The first question that suggests itself is the following: Is it possible to secure a basis Σ_5 for the equation G^5

$$(G^5) \qquad \xi = \eta - zJ^5\kappa\eta$$

of such a nature that the corresponding inner product

$$J^5\alpha\beta$$

shall have as one of its instances the generalized inner product

$$J^4_t J^4_u \alpha(t)\omega(tu)\beta(u),$$

where ω is a function of the class $\mathfrak{K}$? To that end we set

$$J^5\alpha\beta \equiv J^5_{(tu)}\alpha(t)\beta(u),$$

and assign to G^5 the meaning

$$(G^5) \qquad \xi(s) = \eta(s) - zJ^5_{(tu)}\kappa(st)\eta(u) \quad (s).$$

Further, since the operand $\alpha(t)\beta(u)$ belongs to the product class $\mathfrak{M}\mathfrak{M}$, just as we were led to replace J^2 on $\mathfrak{M}^2$ by J^3 and J^4 on $\mathfrak{N} \equiv \mathfrak{M}^2_*$, we stipulate that J^5 shall be a functional operation on the class $(\mathfrak{M}\mathfrak{M})_*$, that is, on the kernel class $\mathfrak{K}$. Thus we have reached the basis

$$\Sigma_5 = (\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M};\ \mathfrak{K} \equiv (\mathfrak{M}\mathfrak{M})_*;\ J),$$

where J is a functional operation on $\mathfrak{K}$ to $\mathfrak{A}$.

This basis Σ_5 is, by the omission of the class $\mathfrak{N} \equiv \mathfrak{M}^2_*$, simpler than the basis Σ_4. Further, the equation G^4 is an instance of the equation G^5, viz., for the operation J^5 with

$$J^5_{(tu)}\varphi(tu) = J^4_t\varphi(tt)$$

for every function φ of $\mathfrak{K}$. This stipulation is legitimate since for every function $\varphi(tu)$ of $\mathfrak{K}_{tu} \equiv (\mathfrak{M}_t\mathfrak{M}_u)_*$ the reduced function $\varphi(tt)$ belongs to $\mathfrak{N}_t \equiv \mathfrak{M}^2_{t*}$. Accordingly, the general theory of the equation G^5 based on Σ_5 yields, as a special instance, a theory of the equation G^4 based on Σ_4.

Furthermore, in accordance with the derivation of the basis Σ_5, the general theory of the equation G^5 contains as an instance a theory of the equation

$$(G^{4\omega}) \qquad \xi(s) = \eta(s) - zJ^4_t J^4_u \kappa(st)\omega(tu)\eta(u), \quad (s),$$

where ω is a function of $\mathfrak{K}$. The operation J^5 for this instance is the operation

$$J^5_{(tu)}\varphi(tu) \equiv J^4_t J^4_u \varphi(tu)\omega(tu)$$

for every function φ of $\mathfrak{K}$. Under the postulates to be laid on the basis Σ_4 the double operation $J^4_t J^4_u$ is applicable to the product $\varphi(tu)\omega(tu)$ of two functions of the class $\mathfrak{K}$.

It is to be noted, however, that *the geometric analogy*, let us say, *between the sphere and the ellipsoid* had as primary function not that of enabling us to treat the equation $G^{4\omega}$

based on Σ_4 as a special instance of the equation G^5 based on Σ_5; at least for the Fredholm theory, the equation $G^{4\omega}$ may be treated as the equation G^4 with the kernel $\kappa(st)$ replaced by $J^4_t\kappa(st)\omega(tu)$, since under the postulates to be imposed on the basis Σ_4 this is a function of the kernel class $\mathfrak{K}$ and the operations J^4_t, J^4_u are commutative. Its primary function was rather to lead us from the basis Σ_4 to a basis Σ_5, possessing in common with Σ_4 the closure property C_1, and possessing furthermore the *closure property* C_2—that for the basis Σ_5 a similar use of the geometric analogy leaves us on the basis Σ_5. In fact, if we seek a basis Σ for the equation G of such a nature that the corresponding inner product

$$J\alpha\beta$$

shall have as one of its instances the generalized inner product

$$J^5_{(tu)}J^5_{(vw)}\alpha(t)\omega(uv)\beta(w),$$

where ω is a function of the class $\mathfrak{K}$, we are led to set

$$J\alpha\beta \equiv J_{(tw)}\alpha(t)\beta(w);$$

thus the new operation J is still an operation J on $\mathfrak{M}\mathfrak{M}$ or preferably on $(\mathfrak{M}\mathfrak{M})_* \equiv \mathfrak{K}$, that is, an operation of precisely the same type as J^5.

As in the preceding remarks, we notice that the equation

$$(G^{5\omega}) \qquad \xi(s) = \eta(s) - zJ^5_{(tu)}J^5_{(vw)}\kappa(st)\omega(uv)\eta(w) \quad (s)$$

may be treated, either as the equation G^5 with the kernel $\kappa(st)$ replaced by $J^5_{(tu)}\kappa(st)\omega(uv)$, or as the equation G^5 with the operation $J^5_{(tu)}$ replaced by a new operation $J'^5_{(tu)}$ defined by the equation

$$J'^5_{(tu)}\varphi(tu) \equiv J^5_{(tw)}J^5_{(vu)}\varphi(tu)\omega(wv)$$

for every function φ of $\mathfrak{K}$.

8. *Resumé.*

To summarize our course to this point: We have as data the four analogous equations (I, II$_n$, III, IV) with their four analogous theories. The theories are of two stages: for brevity, F, the Fredholm theory of the general kernel, and H, the Hilbert-Schmidt theory of the real symmetric or of the more general hermitian kernel.

Guided by the heuristic principle of unification by abstraction, we formulate the general equation

$$(G) \qquad \xi = \eta - zJ_\kappa\eta,$$

embracing as instances the four typical equations; and we seek the foundations, viz., the terminology or basis, and the postulates, of a general theory of the equation G which shall embrace as instances the four theories of at least the regular cases of the respective equations. This is the fundamental closure property C_1.

The form of the equation G dictates the basis Σ_1, which by consideration of the typical equations and of the obvious nature of their theories we simplify to Σ_2, Σ_3, Σ_4 in succession.

The metric-geometric analogy of the sphere and the ellipsoid leads on to the present basis

$$\Sigma_5 = (\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M};\ \mathfrak{K} \equiv (\mathfrak{M}\mathfrak{M})_*;\ J \text{ on } \mathfrak{K} \text{ to } \mathfrak{A}),$$

with the general equation G interpreted as meaning

$$(G) \qquad \xi(s) = \eta(s) - zJ_{(tu)\kappa(st)}\eta(u) \quad (s).$$

And for this basis and its theory there is the closure property C_2.

The operation J is a functional operation on $\mathfrak{K}$ to $\mathfrak{A}$, that is, if φ is a function of the class $\mathfrak{K}$,

$$J\varphi \equiv J_{(tu)}\varphi(tu)$$

denotes a definite number a of the class $\mathfrak{A}$. For purposes of application J or $J_{(tu)}$ is often definable as a double operation, in the form $J_{(tu)} \equiv J_t'J_u''$. For purposes of the general theory of the basis Σ_5, however, $J_{(tu)}$ is an *indivisible operation*. Of course $J_{(tu)}\varphi(tu)$ is equal, e. g., to $J_{(tw)}\varphi(tw)$ or $J_{(vu)}\varphi(vu)$ or $J_{(vw)}\varphi(vw)$, but it is in general not equal to $J_{(ut)}\varphi(tu)$.

9. *The Definitive Basis* Σ_6.

The basis Σ_5 is in effect definitive for the general Hilbert-Schmidt theory H. Suitable postulates will be given for both bases Σ_4, Σ_5. However, for the general Fredholm theory F the metric-geometric analogy, by leading us from the operation J_t on $\mathfrak{R}_t$ over to the operation $J_{(tu)}$ on $\mathfrak{K}_{tu}$, enables us to proceed to a still more general basis Σ_6.

In fact, for the theory H we have the respective definitions

$$\kappa(ts) = \kappa(st) \quad \text{and} \quad \kappa(ts) = \overline{\kappa(st)},$$

of the symmetry of the real-valued kernel κ and the hermitian character of the complex-valued kernel κ. These definitions require the arguments s and t to have the same range $\mathfrak{P}$.

There is, however, no such necessity in the theory F. Thus we are led to the following definitive basis for the theory F, viz.,

$$\Sigma_6 = \left(\mathfrak{A};\ \begin{matrix}\tilde{\mathfrak{P}} & \tilde{\mathfrak{M}} & \mathfrak{K} \equiv (\tilde{\mathfrak{M}}\hat{\mathfrak{M}})_* \\ \hat{\mathfrak{P}}; & \hat{\mathfrak{M}}; & \check{\mathfrak{K}} \equiv (\hat{\mathfrak{M}}\tilde{\mathfrak{M}})_*\end{matrix};\ J \text{ on } \check{\mathfrak{K}} \text{ to } \mathfrak{A}\right).$$

Here $\tilde{\mathfrak{P}}$, $\hat{\mathfrak{P}}$ are two conceptually but not necessarily actually distinct classes of elements $\tilde{p}$, $\hat{p}$ respectively; $\tilde{\mathfrak{M}}$, $\hat{\mathfrak{M}}$ are two classes of functions $\tilde{\mu}$ on $\tilde{\mathfrak{P}}$, $\hat{\mu}$ on $\hat{\mathfrak{P}}$ respectively; $\mathfrak{K}$ and $\check{\mathfrak{K}}$ are the $*$-composites of the two classes $\tilde{\mathfrak{M}}$, $\hat{\mathfrak{M}}$ in the respective orders $\tilde{\mathfrak{M}}$, $\hat{\mathfrak{M}}$; $\hat{\mathfrak{M}}$, $\tilde{\mathfrak{M}}$—that is, the functions κ of $\mathfrak{K}$ bear their arguments $\tilde{p}$, $\hat{p}$ in the order $(\tilde{p}\hat{p})$, while the functions $\check{\kappa}$ of $\check{\mathfrak{K}}$ bear their arguments in the reverse order $(\hat{p}\tilde{p})$; thus there is a correspondence of the functions κ, $\check{\kappa}$ of such a nature that for two corresponding functions κ, $\check{\kappa}$ we have $\kappa(\tilde{p}\hat{p}) = \check{\kappa}(\hat{p}\tilde{p})$ for every pair of arguments $\tilde{p}$ of $\tilde{\mathfrak{P}}$, $\hat{p}$ of $\hat{\mathfrak{P}}$; and functions κ, $\check{\kappa}$ occurring together are understood to be in this sense corresponding functions, each the *transpose* of the other. Finally, J is a functional operation on $\check{\mathfrak{K}}$ to $\mathfrak{A}$, so that for every function $\check{\kappa}$ of $\check{\mathfrak{K}}$ the expression

$$J\kappa \equiv J_{(tu)}\check{\kappa}(tu),$$

denotes a definite number a of $\mathfrak{A}$. Here, in order to make the formulas based on Σ_6 readily comparable with those based on Σ_5, we agree that the elements

$$\tilde{p},\ s,\ u,\ w;$$

$$\hat{p},\ t,\ v,\ r,$$

are generic elements of the classes $\tilde{\mathfrak{P}}$; $\hat{\mathfrak{P}}$ respectively.

Based on Σ_6 we have the general pair of adjoint equations

$$(G) \qquad \xi = \tilde{\eta} - zJ\kappa\tilde{\eta};$$

$$(G) \qquad \xi = \hat{\eta} - zJ\hat{\eta}\kappa,$$

with the meanings

$$(G) \qquad \tilde{\xi}(s) = \tilde{\eta}(s) - zJ_{(tu)}\kappa(st)\eta(u) \quad (s);$$

$$(\breve{G}) \qquad \hat{\xi}(t) = \hat{\eta}(t) - zJ_{(rs)}\hat{\eta}(r)\kappa(st) \quad (t).$$

Introducing the functional operation $\breve{J}$, the *transpose* of J, by the equation: $\breve{J}\kappa = J\breve{\kappa}$, for every function κ of $\mathfrak{K}$, so that $\breve{J}$ is on $\mathfrak{K}$ to $\mathfrak{A}$, we have the adjoint equation $\breve{G}^6$ in the form

$$(\breve{G}) \qquad \hat{\xi} = \hat{\eta} - z\breve{J}\breve{\kappa}\hat{\eta},$$

viz., in the form G for the transpose basis $\breve{\Sigma}_6$, which is the basis Σ_6 with the interchange of rôles of $\tilde{\mathfrak{P}}$, $\hat{\mathfrak{P}}$; $\tilde{\mathfrak{M}}$, $\hat{\mathfrak{M}}$; $\mathfrak{K}$, $\breve{\mathfrak{K}}$; J, $\breve{J}$.

The basis Σ_5 is secured from the basis Σ_6 by the supposition

$$\tilde{\mathfrak{P}} = \hat{\mathfrak{P}} \equiv \mathfrak{P}; \quad \tilde{\mathfrak{M}} = \hat{\mathfrak{M}} \equiv \mathfrak{M}.$$

Thus the theory for the basis Σ_6 has the closure property C_1. It has moreover the closure property C_2, on the understanding that the function ω or $\omega(uv)$ is any function of the class $\mathfrak{K}$; that is, the functional operation J'

$$J'_{(tu)}\varphi(tu) \equiv J^6_{(tw)}J^6_{(vu)}\varphi(tu)\omega(wv)$$

for every function φ of the class $\breve{\mathfrak{K}}$, is a functional operation of the type J^6 on $\breve{\mathfrak{K}}$ to $\mathfrak{A}$. Further, as on the basis Σ_5, the equation

$$(G^{6\omega}) \qquad \tilde{\xi}(s) = \tilde{\eta}(s) - zJ^6_{(tu)}J^6_{(vw)}\kappa(st)\omega(uv)\tilde{\eta}(w) \quad (s)$$

may be treated, either as the equation G^6 with the kernel $\kappa(st)$ replaced by $J^6_{(tw)}\kappa(st)\omega(uv)$, or as the equation G^6 with the operation J^6 replaced by the operation J' defined above.

Setting

$$\kappa\begin{pmatrix} s_1, & \cdots, & s_n \\ t_1, & \cdots, & t_n \end{pmatrix} \equiv |\,\kappa(s_i t_j)\,| \quad (i, j = 1, \cdots, n),$$

we define the Fredholm determinant $F_\kappa(z)$ of the kernel κ and the parameter z, for the general theory of the adjoint equations G, $\breve{G}$ based on Σ_6, as follows:

$$F_\kappa(z) \equiv \sum_{k=0}^{\infty} \frac{(-z)^k}{k!} J_{(t_1 s_1)} \cdots {}_{(t_k s_k)}\, \kappa\begin{pmatrix} s_1, & \cdots, & s_k \\ t_1, & \cdots, & t_k \end{pmatrix}.$$

Here $J_{(t_1s_1)} \cdots {}_{(t_hs_h)}$ denotes the k-fold operation

$$J_{(t_1s_1)}J_{(t_2s_2)} \cdots J_{(t_ks_k)}.$$

The hth minor ($h = 1, 2, \cdots$) has the definition

$$F_\kappa\begin{pmatrix} s_1, & \cdots, & s_h \\ t_1, & \cdots, & t_h \end{pmatrix}; z\Big)$$

$$\equiv (-1)^h \sum_{k=0}^{\infty} \frac{(-z)^k}{k!} J_{(t_{h+1}s_{h+1})\cdots(t_{h+k}s_{h+k})} \kappa\begin{pmatrix} s_1, & \cdots, & s_{h+k} \\ t_1, & \cdots, & t_{h+k} \end{pmatrix}$$

The initial terms ($k = 0$) of the determinant and of the hth minor are respectively

$$1; \quad (-1)^h \kappa\begin{pmatrix} s_1, & \cdots, & s_h \\ t_1, & \cdots, & t_h \end{pmatrix}.$$

Under the postulates to be specified below, Hadamard's theorem on determinants may be utilized to show that these series are permanently convergent power series in z, as to the hth minor for all values of the arguments $s_1, \cdots, t_h$. Further, on every finite circle in the z-plane the hth minor series converges uniformly over the composite range $\mathfrak{P}_1 \cdots \mathfrak{P}_h$ relative to the class

$$\mathfrak{K}_{s_1 \cdots t_h} \equiv (\mathfrak{M}_{s_1} \cdots \mathfrak{M}_{t_h})_*;$$

and its terms belong to the class $\mathfrak{K}_{s_1 \cdots t_h}$; and, as this class is closed, the hth minor for every z as a function of the arguments $s_1, \cdots, t_h$ belongs to the class $\mathfrak{K}_{s_1 \cdots t_h}$.

The general theory of the adjoint equations G, $\breve{G}$ proceeds along the usual lines. Thus, if the parameter z is not a root of the determinant $F_\kappa(z)$, the kernel κ has the reciprocal kernel λ

$$\lambda(st) \equiv \frac{F_\kappa\begin{pmatrix} s \\ t \end{pmatrix}; z\Big)}{F_\kappa(z)},$$

belonging to the class $\mathfrak{K}$ and satisfying the equations

$$\kappa(st) + \lambda(st) = zJ_{(vw)}\kappa(sv)\lambda(wt) = zJ_{(vw)}\lambda(sv)\kappa(wt) \quad (st);$$

and the equations G, $\breve{G}$ have the solutions

$$\tilde{\eta} = \tilde{\xi} - zJ\lambda\tilde{\xi}, \quad \hat{\eta} = \hat{\xi} - zJ\hat{\xi}\lambda.$$

10. *Adjunctional Composition. The Closure Property C_3.*

In the algebraic domain we proceed from the single equation I

$$x = y - zky \tag{I}$$

to the system II_n of n simultaneous equations

$$x_i = y_i - z\sum_{j=1}^{n} k_{ij}y_j \qquad (i = 1, \cdots, n). \tag{II_n}$$

Similarly, on the basis Σ_6, we proceed from the single equation G

$$\tilde{\xi}(s) = \tilde{\eta}(s) - zJ_{(tu)}\kappa(st)\tilde{\eta}(u) \quad (s) \tag{G}$$

to the system G_n of n simultaneous equations

$$\xi_i(s) = \tilde{\eta}_i(s) - z\sum_{j=1}^{n} J_{(tu)}\kappa_{ij}(st)\eta_j(u) \quad (si). \tag{G_n}$$

Here we have given the parameter z, the n^2 functions κ_{ij} of the kernel class $\mathfrak{K}$, and the n functions $\tilde{\xi}_i$ of the class $\tilde{\mathfrak{M}}$; and we are to determine the n functions $\tilde{\eta}_i$ of the class $\tilde{\mathfrak{M}}$. For the instance IV on the linear interval a–b Fredholm showed how to reduce the system IV_n to a single equation IV on the linear interval a–b_n $(b_n = a + n(b - a))$. A similar procedure is effective to reduce the system G_n on the basis Σ_6 to a single equation G on an enlarged basis. The procedure, however, is capable of further generalization, and, as thus generalized, of an important application to the theory of mixed linear equations.

To this end, consider a system of n bases Σ_6^i $(i = 1, \cdots, n)$, viz.,

$$\Sigma_6^i = \left(\mathfrak{A};\ \begin{matrix}\tilde{\mathfrak{P}}^i & \tilde{\mathfrak{M}}^i & \mathfrak{K}^i \equiv (\tilde{\mathfrak{M}}^i\tilde{\mathfrak{M}}^i)\\ \hat{\mathfrak{P}}^i; & \hat{\mathfrak{M}}^i; & \check{\mathfrak{K}}^i \equiv (\hat{\mathfrak{M}}^i\check{\mathfrak{M}}^i)\end{matrix};\ J^i \text{ on } \check{\mathfrak{K}}^i \text{ to } \mathfrak{A}\right).$$

The class $\mathfrak{A}$ is the same in the n bases Σ^i; otherwise the bases are conceptually but not necessarily actually distinct; it is however convenient to suppose that no two of the n classes $\tilde{\mathfrak{P}}^i$ have common elements and that no two of the n classes $\hat{\mathfrak{P}}^i$ have common elements; this state of affairs, being always securable by transformation, involves no restriction of generality.

As a generalization of the system G_n on the basis Σ_6 we have on the system of n bases Σ_6^i the system $G_n^{1\cdots n}$

$$(G_n^{1\cdots n}) \quad \xi^i(s^i) = \tilde{\eta}^i(s^i) - z\sum_{j=1}^{n} J^j_{(s^i u^j)}\kappa^{ij}(s^i t^j)\tilde{\eta}^j(u^j) \quad (s^i i),$$

of n simultaneous equations. Here we have given the parameter z, the n functions ξ^i of the respective classes $\mathfrak{M}^i$, and the n^2 functions κ^{ij} of the respective classes $\mathfrak{K}^{ij} \equiv (\mathfrak{M}^i\mathfrak{M}^j)_*$, so that $\mathfrak{K}^{ii} = \mathfrak{K}^i$; and we are to determine the n functions $\tilde{\eta}^i$ of the respective classes $\mathfrak{M}^i$. The n equations ($i = 1, \cdots, n$) of the system $G_n^{1\cdots n}$ are to hold for every value of the respective arguments s^i of the class $\mathfrak{P}^i$.

We impose on each of the n bases Σ_6^i the postulates to be specified below. Then the *adjunctional composite* $\Sigma_6^{1\cdots n}$ of the n bases Σ_6^i is a basis Σ_6 satisfying the same postulates, and the system $G_n^{1\cdots n}$ of n simultaneous equations is equivalent to a single equation G on the composite basis Σ_6. This is the *closure property* C_3 of the theory of the linear equation G on the basis Σ_6.

For the adjunctional composite $\Sigma_6^{1\cdots n}$ of the n bases Σ_6^i the class $\mathfrak{A}$ is the common class $\mathfrak{A}$ of the *component bases* Σ_6^i. The ranges $\tilde{\mathfrak{P}}$, $\hat{\mathfrak{P}}$ are the *adjunctional composites* or logical sums respectively of the ranges $\tilde{\mathfrak{P}}^i$, $\hat{\mathfrak{P}}^i$; the range $\tilde{\mathfrak{P}}^i$ is the ith *component* of the range $\tilde{\mathfrak{P}}$, and the range $\hat{\mathfrak{P}}^i$ is the ith *component* of the range $\hat{\mathfrak{P}}$. A function $\tilde{\theta}$ on the range $\tilde{\mathfrak{P}}$ determines n *component functions* $\tilde{\theta}^i$ on the respective component ranges $\tilde{\mathfrak{P}}^i$; and, conversely, any n functions $\tilde{\theta}^i$ on the respective ranges $\tilde{\mathfrak{P}}^i$ are the n components of a definite function $\tilde{\theta}$ on the composite range $\tilde{\mathfrak{P}}$; this function $\tilde{\theta}$ is the *adjunctional composite* of the n functions $\tilde{\theta}^i$. The functions $\tilde{\mu}$ on the range $\tilde{\mathfrak{P}}$ obtained thus by adjunctional composition of n functions $\tilde{\mu}^i$ of the respective classes $\tilde{\mathfrak{M}}^i$ on the ranges $\tilde{\mathfrak{P}}^i$ constitute the *adjunctional composite* $\tilde{\mathfrak{M}}$ of the n classes $\tilde{\mathfrak{M}}^i$. The classes $\tilde{\mathfrak{M}}$, $\hat{\mathfrak{M}}$ of functions $\tilde{\mu}$, $\hat{\mu}$ on the composite ranges $\tilde{\mathfrak{P}}$, $\hat{\mathfrak{P}}$ of the composite basis $\Sigma_6^{1\cdots n}$ are the adjunctional composites, in this sense, respectively of the classes $\tilde{\mathfrak{M}}^i$, $\hat{\mathfrak{M}}^i$ of functions $\tilde{\mu}^i$, $\hat{\mu}^i$ on the component ranges $\tilde{\mathfrak{P}}^i$, $\hat{\mathfrak{P}}^i$. The product ranges $\tilde{\mathfrak{P}}\tilde{\mathfrak{P}}$, $\hat{\mathfrak{P}}\tilde{\mathfrak{P}}$ are the adjunctional composites respectively of the n^2 product

ranges $\mathfrak{P}^i\mathfrak{P}^j$, $\mathfrak{P}^i\overline{\mathfrak{P}}^j$. Then, under the postulates, the classes $\mathfrak{K} \equiv (\mathfrak{M}\mathfrak{M})_*$, $\breve{\mathfrak{K}} \equiv (\mathfrak{M}\overline{\mathfrak{M}})_*$ of functions κ, $\breve{\kappa}$ on the ranges $\mathfrak{P}\mathfrak{P}$, $\mathfrak{P}\overline{\mathfrak{P}}$ are the adjunctional composites respectively of the classes $\mathfrak{K}^{ij} \equiv (\mathfrak{M}^i\mathfrak{M}^j)_*$, $\breve{\mathfrak{K}}^{ij} \equiv (\mathfrak{M}^i\overline{\mathfrak{M}}^j)_*$ of functions κ^{ij}, $\breve{\kappa}^{ij}$ on the component product ranges $\mathfrak{P}^i\mathfrak{P}^j$, $\mathfrak{P}^i\overline{\mathfrak{P}}^j$. The functional operation J on the class $\breve{\mathfrak{K}}$ of the composite basis $\Sigma_6^{1\cdots n}$ is the *adjunctional composite* of the n functional operations J^i on the respective classes $\breve{\mathfrak{K}}^i$ or $\breve{\mathfrak{K}}^{ii}$ of the component bases Σ^i, viz., if the function $\breve{\kappa}$ is the adjunctional composite of the functions $\breve{\kappa}^{ij}$,

$$J\breve{\kappa} \equiv \sum_{i=1}^{n} J^i\breve{\kappa}^{ii}.$$

Now we see readily that the linear equation

$$(G) \qquad \tilde{\xi} = \tilde{\eta} - zJ\kappa\tilde{\eta}$$

for the composite basis $\Sigma_6^{1\cdots n}$, and the system

$$(G_n^{1\cdots n}) \qquad \tilde{\xi}^i = \tilde{\eta}^i - z\sum_{j=1}^{n} J^j\kappa^{ij}\tilde{\eta}^j \qquad (i = 1, \cdots, n)$$

of n simultaneous equations for the system of component bases Σ_6^i, are precisely equivalent. The functions $\tilde{\xi}$, $\tilde{\eta}$, κ are the adjunctional composites respectively of the functions $\tilde{\xi}^i$, $\tilde{\eta}^i$, κ^{ij}.

Accordingly, the theory of the equation G based on Σ_6 covers the theory of the system $G_n^{1\cdots n}$ based on the Σ_6^i. Thus, if the parameter z is not a root of the Fredholm determinant $F_\kappa(z)$, the kernel κ, the composite of the functions κ^{ij}, has a reciprocal kernel λ, the composite of certain functions λ^{ij}, and, corresponding to the equations

$$\kappa + \lambda = zJ\kappa\lambda = zJ\lambda\kappa,$$

we have the system of equations

$$\kappa^{ij} + \lambda^{ij} = z\sum_{k=1}^{n} J^k\kappa^{ik}\lambda^{kj} = z\sum_{k=1}^{n} J^k\lambda^{ik}\kappa^{kj} \quad (ij),$$

and for the system $G_n^{1\cdots n}$ we have the solution

$$\tilde{\eta}^i = \tilde{\xi}^i - z\sum_{j=1}^{n} J^j\lambda^{ij}\tilde{\xi}^j \qquad (i=1, \cdots, n).$$

For the adjoint system of equations

$$(\breve{G}_n^{1\cdots n}) \qquad \hat{\xi}^j = \hat{\eta}^j - z\sum_{i=1}^{n} J^i\hat{\eta}^i\kappa^{ij} \qquad (j = 1, \cdots, n)$$

we have, with the same functions λ^{ij}, the solution

$$\hat{\eta}^j = \hat{\xi}^j - z\sum_{i=1}^{n} J^i\hat{\xi}^i\lambda^{ij} \qquad (j = 1, \cdots, n).$$

The Mixed Linear Equation.—Consider a basis Σ_6 with n functional operations $J_1, \cdots, J_n$ (instead of merely one) on the class $\breve{\mathfrak{R}}$, and the corresponding mixed linear equation

$$\tilde{\xi} = \tilde{\eta} - z\sum_{j=1}^{n} J_j\kappa_j\tilde{\eta}.$$

Here the function $\tilde{\xi}$ of the class $\tilde{\mathfrak{M}}$ and n functions κ_j of the kernel class $\mathfrak{K}$ are given, and the function η of the class $\tilde{\mathfrak{M}}$ is to be found.

We may treat this mixed basis as a system of n bases Σ_6^i, identical except that the functional operations J^i of the bases Σ^i are the respective operations J_i of the mixed basis Σ_6. Then the mixed linear equation n times repeated constitutes a particular case of the system $G_n^{1\cdots n}$ of n simultaneous equations on the system of bases Σ_6^i. Accordingly, if the parameter z is not a root of the Fredholm determinant, the system of kernel functions κ_j has, with respect to the system of functional operations J_j, a reciprocal system of kernel functions λ_j; and we have the equations

$$\kappa_j + \lambda_j = z\sum_{k=1}^{n} J_k\kappa_k\lambda_j = z\sum_{k=1}^{n} J_k\lambda_k\kappa_j \quad (j),$$

and for the mixed equation the solution

$$\tilde{\eta} = \tilde{\xi} - z\sum_{j=1}^{n} J_j\lambda_j\tilde{\xi}.$$

11. **-Composition. The Closure Property C_4.*

We have seen that the theory of the general linear equation G based on Σ_6 has (closure property C_1) as instances the theories of the equations (I, II_n, III, IV), and furthermore (closure property C_3) as instances the theories of the systems G_n, $G_n^{1\cdots n}$ of linear equations based respectively on Σ_6 and on a system of bases Σ_6^i. Now the step from G on Σ_6 to G_n on Σ_6 was analogous to and has as instance the step from I to II_n. The question arises whether the general theory has as instances the theories of equations arising from G by steps

analogous to the steps from I to III, IV, and even to the general equation G based on Σ_6. Under the postulates to be specified, this question is to be answered in the affirmative, and this is the *closure property* C_4 of the general theory.

Consider again a system of n bases Σ_6^i having the same class $\mathfrak{A}$ of all real or of all complex numbers, and otherwise conceptually, but not necessarily actually, distinct. From this system of bases Σ^i we determine the *$*$-composite basis* $\Sigma^{1\cdots n*}$ or Σ^* as follows. The class $\mathfrak{A}$ of Σ^* is the common class $\mathfrak{A}$ of the *constituent* bases Σ^i. The ranges $\tilde{\mathfrak{P}}$, $\hat{\mathfrak{P}}$ of Σ^* are the product classes respectively of the ranges $\tilde{\mathfrak{P}}^i$, $\hat{\mathfrak{P}}^i$ of the bases Σ^i. The classes $\tilde{\mathfrak{M}}$, $\hat{\mathfrak{M}}$ of functions on $\tilde{\mathfrak{P}}$, $\hat{\mathfrak{P}}$ are the $*$-composites respectively of the classes $\tilde{\mathfrak{M}}^i$, $\hat{\mathfrak{M}}^i$ on $\tilde{\mathfrak{P}}^i$, $\hat{\mathfrak{P}}^i$ of the bases Σ^i. Then the product ranges $\tilde{\mathfrak{P}}\hat{\mathfrak{P}}$, $\hat{\mathfrak{P}}\tilde{\mathfrak{P}}$ of Σ^* are the product classes respectively of the product ranges $\tilde{\mathfrak{P}}^i\hat{\mathfrak{P}}^i$, $\hat{\mathfrak{P}}^i\tilde{\mathfrak{P}}^i$ of the bases Σ^i; and the classes, $\mathfrak{K} \equiv (\tilde{\mathfrak{M}}\hat{\mathfrak{M}})_*$, $\check{\mathfrak{K}} \equiv (\hat{\mathfrak{M}}\tilde{\mathfrak{M}})_*$ of functions κ, $\check{\kappa}$ on $\tilde{\mathfrak{P}}\hat{\mathfrak{P}}$, $\hat{\mathfrak{P}}\tilde{\mathfrak{P}}$ are the $*$-composites respectively of the classes $\mathfrak{K}^i \equiv (\tilde{\mathfrak{M}}^i\hat{\mathfrak{M}}^i)_*$, $\check{\mathfrak{K}}^i \equiv (\hat{\mathfrak{M}}^i\tilde{\mathfrak{M}}^i)_*$ of functions κ^i, $\check{\kappa}^i$ on $\tilde{\mathfrak{P}}^i\hat{\mathfrak{P}}^i$, $\hat{\mathfrak{P}}^i\tilde{\mathfrak{P}}^i$ of the bases Σ^i. The functional operation J on the class $\check{\mathfrak{K}}$ has the definition

$$J\check{\kappa} \equiv J^1 \cdots J^n\check{\kappa}$$

for every function $\check{\kappa}$ of the class $\check{\mathfrak{K}}$.

This $*$-composite basis Σ^* satisfies the postulates laid on the bases Σ_6, and accordingly the general theory of the equation G based on Σ_6 has as instance the theory of the equation G for the basis Σ^*, that is, of the equation

$$\tilde{\xi}(s^1\cdots s^n) = \tilde{\eta}(s^1\cdots s^n) - zJ^1_{(t^1u^1)}\cdots J^n_{(t^nu^n)}\kappa(s^1\cdots t^n)\tilde{\eta}(u^1\cdots u^n) \quad (s^1\cdots s^n).$$

The $*$-composition of the bases II_n and Σ_6 leads to the equation

$$\tilde{\xi}(is) = \tilde{\eta}(is) - z\sum_{j=1}^{n} J_{(tu)}\kappa(isjt)\tilde{\eta}(ju) \quad (is),$$

which is, notation apart, the system G_n on the basis Σ_6. In fact, the $*$-composite of II_n and Σ_6 is identical with the adjunctional composite of n systems identical with Σ_6. But the general adjunctional composition of n systems possibly distinct is not an instance of the $*$-composition here defined.

12. *Additional Definitions.*

We are to specify postulates on the bases Σ_4, Σ_5, Σ_6 enabling us to secure general theories F, H. To that end we have need of several additional definitions.

Consider a class $\mathfrak{M}$ of functions μ on the range $\mathfrak{P}$ to $\mathfrak{A}$. We have already defined the properties *linearity* (L), *closure* (C), and now define two *dominance properties* (D, D_0) and a *reality property* (R).

The function α is dominated by the function β in case for every argument p $|\alpha(p)| \leqq |\beta(p)|$. The class $\mathfrak{M}$ has the dominance property D_0 in case every function μ of $\mathfrak{M}$ is dominated by some real-valued nowhere negative function μ_0 of $\mathfrak{M}$; the function μ_0 may vary with μ. The class $\mathfrak{M}$ has the dominance property D in case for every finite or infinite sequence $\{\mu_n\}$ of functions of $\mathfrak{M}$ there is a function μ of $\mathfrak{M}$ such that the functions μ_n of the sequence $\{\mu_n\}$ are dominated respectively by certain numerical multiples $a_n\mu$ of the function μ, that is, for every n and p $|\mu_n(p)| \leqq |a_n\mu(p)|$.

A complex number a has a conjugate complex number $\bar{a}$. A function α has a conjugate function $\bar{\alpha}$, whose functional values $\bar{\alpha}(p)$ are conjugate to the corresponding functional values $\alpha(p)$, of the function α. A class $\mathfrak{M}$ of functions μ has a conjugate class $\mathfrak{M}$ consisting of the functions $\bar{\mu}$ conjugate to the various functions μ of the class $\mathfrak{M}$. A number a is real if $a = \bar{a}$. A function α is real or real-valued if $\alpha = \bar{\alpha}$. Similarly, a class $\mathfrak{M}$ is *real* (R) if $\mathfrak{M} = \bar{\mathfrak{M}}$. Thus, the class of complex-valued continuous functions on a linear interval is real. In general, a real linear class of functions is a linear class of real-valued functions or a class of complex-valued functions whose constituent functions have real and imaginary components which range independently over a linear class of real-valued functions.

Consider a functional operation J on a class $\mathfrak{M}$ of functions μ. The operation J is *linear* (L) in case $\mu = a_1\mu_1 + a_2\mu_2$ implies $J\mu = a_1J\mu_1 + a_2J\mu_2$. The operation J has the *modular property* (M) in case there exists an *associated modulus* M, viz., a functional operation M on real-valued nowhere negative functions μ_0 of $\mathfrak{M}$ such that $M\mu_0$ is a real non-negative number, for which a relation $|\mu(p)| \leqq \mu_0(p)$ holding for every p implies the relation $|J\mu| \leqq M\mu_0$.

13. *Postulates for the Theory F.*

We secure the general Fredholm theory F of the adjoint equations G, $\breve{G}$ based on the respective systems Σ_4; Σ_5; Σ_6 by postulating that the respective classes $\mathfrak{M}$; $\mathfrak{M}$; $\breve{\mathfrak{M}}$, $\hat{\mathfrak{M}}$ have the properties $L\ C\ D\ D_0$, and that the functional operation J on $\mathfrak{N} \equiv \mathfrak{M}_*^2$; $\mathfrak{K} \equiv (\mathfrak{M}\mathfrak{M})_*$; $\breve{\mathfrak{K}} \equiv (\hat{\mathfrak{M}}\breve{\mathfrak{M}})_*$ has the properties $L\ M$; and for this theory we have the four closure properties $C_1\ C_2\ C_3\ C_4$; the theory F based on Σ_4 however lacks the closure property C_2.

14. *Postulates for the Theory H.*

We secure the general Hilbert-Schmidt theory H for the complex-valued hermitian kernels $\kappa(\bar{\kappa} = \breve{\kappa})$ based on the system Σ_4 or Σ_5, by postulating that the class $\mathfrak{A}$ is the class of all complex numbers, that the class $\mathfrak{M}$ has the properties $L\ C\ D\ D_0\ R$, and that the functional operation J on the class $\mathfrak{N} \equiv \mathfrak{M}_*^2$ or $\mathfrak{K} \equiv (\mathfrak{M}\mathfrak{M})_*$ has the properties $L\ M\ H\ P\ P_0$. The operation J is *hermitian* (H) in case for every two functions α, β of $\mathfrak{M}$ $\overline{J\alpha\beta} = J\overline{\beta\alpha}$, from which, in view of the properties of $\mathfrak{M}$ and the properties $L\ M$ of J, follows the relation $\overline{J\nu} = J\overline{\nu}$ or $\overline{J\kappa} = J\breve{\kappa}$ for every function ν of $\mathfrak{N}$ or κ of $\mathfrak{K}$. The operation J is *definitely positive* $(P\ P_0)$ in case for every function μ of $\mathfrak{M}$ the result $J\mu\bar{\mu}$ (for a hermitian operation J necessarily a real number) is (P) a real non-negative number (P_0) vanishing only if $\mu = 0$. Thus the operation J may be described as a *definitely positive* (PP_0) *linear* (L) *hermitian* (H) *operation having* (M) *an associated modular operation* M.

The general theory H based on the system Σ_4 or Σ_5 has the four or three closure properties $C_1\ C_3\ C_4$ or $C_1\ C_2\ C_3\ C_4$, where in C_2 the function ω of $\mathfrak{K}$ is hermitian $(\bar{\omega} = \breve{\omega})$ and positively definite, viz., for every function μ of $\mathfrak{M}$ $J_{(rs)}J_{(tu)}\mu(r)\omega(st)\bar{\mu}(u)$ is a non-negative real number vanishing only if $\mu = 0$.

If we postulate that the class $\mathfrak{A}$ is the class of all real numbers, we secure the general Hilbert-Schmidt theory H for (real) symmetric kernels $\kappa(\kappa = \breve{\kappa})$ based on the system Σ_4 or Σ_5, by imposing the same conditions as before on the class $\mathfrak{M}$ and the operation J. However there are certain simplifications. Since all the functions and the operations J are real-valued, the class $\mathfrak{M}$ is necessarily real (R); and the hermitian property (H) of J is the *symmetry* (S) $J\alpha\beta = J\beta\alpha$, holding necessarily

for the theory based on Σ_4, and for the theory based on Σ_5, implying $J_\kappa = J_{\breve{\kappa}}$; and the property (PP_0) of being a definitely positive operation J is that $J_{\mu\mu}$ is (P) a real non-negative number (P_0) vanishing only if $\mu = 0$.

We have specified the bases or terminologies and the postulates of the general theories F and H, and conclude this address on the foundations of the theory of linear integral equations with the expression of grateful appreciation of your so prolonged attention.

THE UNIVERSITY OF CHICAGO.

SHORTER NOTICES.

Lectures on Fundamental Concepts of Algebra and Geometry. By J. W. YOUNG. Prepared for publication with the cooperation of W. W. DENTON, with a Note on the Growth of Algebraic Symbolism by U. G. MITCHELL. New York, The Macmillan Company, 1911. vii + 247 pp.

THE book contains twenty-one lectures on the logical foundations of algebra and geometry in substantially the same form as delivered at the University of Illinois during the summer of 1909, with an appended note on the growth of algebraic symbolism. "The points of view developed and the results reached are not directly of use in elementary teaching. They are extremely abstract, and will be of interest only to mature minds. They should serve to clarify the teacher's ideas and thus indirectly serve to clarify the pupil's." "The results nevertheless, have a direct bearing on some of the pedagogical problems confronting the teacher." "Let the teacher be vitally, enthusiastically interested in what he is teaching, and it will be a dull pupil who does not catch the infection. It is hoped these lectures may give a new impetus to the enthusiasm of those teachers who have not as yet considered the logical foundations of mathematics." Such is the purpose of the author.

The first five lectures, of 57 pages, form an introduction which makes clear the nature of the problems to be discussed and the point of view from which they are approached. Euclid's Elements, a non-euclidian geometry, the history of the parallel postulate, the logical significance of definitions,

axioms and postulates, and the consistency, independence and categoricalness of assumptions are here discussed. The example of a non-euclidean world, which is that of Poincaré in more detailed form, is a forceful means of showing the relation of our intuitional knowledge of space to an abstract geometry. The qualities of consistency, independence, and categoricalness receive emphasis.

The abstract but clear development of the cardinal, negative, rational, irrational, and complex numbers takes up several lectures. The notions of class, order, correspondence, group, variable, function, and limit are carefully developed and discussed. The lecture on limits deserves the special attention of geometry teachers because of its modern point of view. The axioms of Hilbert and Pieri are discussed in considerable detail. The discussion of spaces of four or more dimensions will be of interest in the light of recent popular papers published on the fourth dimension.

Throughout the lectures the historical development of the concepts considered is emphasized; this method of presentation illustrates the importance of the history of a subject to a teacher, for it shows that mathematics is a live and growing science and not fixed and finally determined. Professor G. A. Miller (*Science*, July 7, 1911) has called attention to two slight historical errors. Remarks on pedagogical principles found scattered through the book make it clear that the author is fully aware of the limitations to the use of purely abstract methods in elementary teaching. "No *formal* proof of any proposition should be attempted which seems obvious to the pupil without proof." "With all our insistence on the formal logical procedure, the important fact must not be lost sight of that formal logic is in only a small minority of cases the method of mathematical discovery. Imagination, geometric intuition, experimentation, analogies sometimes of the vaguest sort, and judicious guessing are instruments continually employed in mathematical research." Such statements show that the author retains a proper perspective while emphasizing the value of the abstract method. The following definition of mathematics forms the climax reached in the last lecture: "A mathematical science is any body of propositions which is capable of an abstract formulation and arrangement in such a way that every proposition after a certain one is a formal logical consequence of some or

all the preceding propositions. Mathematics consists of all such mathematical sciences." The appended note treats from a historical point of view the three stages of rhetorical, syncopated, and symbolic algebraic notation.

The book is strongly commended to teachers and prospective teachers and will be very useful to those giving teachers' courses. While a few discussions, such as that of the first three known infinite cardinal numbers, may not be appreciated by those whose knowledge of mathematics is limited, yet practically no use is made of the technique of higher mathematics. The ability for abstract thinking required of the reader is considerable, but the careful reading of such a book cannot fail to have a broadening effect particularly upon those teaching elementary mathematics and in daily contact with immature minds. There is a need for scholarly contributions to mathematical literature in English sufficiently elementary in character to be useful and inspiring to progressive secondary teachers, and we believe the book under review to be just such a contribution.

ERNEST B. LYTLE.

Leçons de Cristallographie. Par G. FRIEDEL. Paris, Hermann, 1911. 8vo. vi + 310 pp. 10 fr.

Two things have been kept in mind by the author in writing this book. One is the purely utilitarian view of furnishing some knowledge of the subject to students of the mining college. The other is the purely cultural view of the subject. The latter might seem a little strange, considering the title. But from a mathematical standpoint, there is a distinct cultural value in the study of geometrical properties in crystallographic form. This becomes evident when we remember the thirty-two types of crystals and the related finite groups. We can assert from experience, having tried the experiment for several years, with freshman classes, that such a study is fully worth the time put upon it, even worth more than some other branches of elementary mathematics. After the class has drawn all the types of crystals, and manufactured one or more apiece, considering in each case the symmetry involved, the rotations possible, the way each face is produced from the original one, they are in a position to know something of group theory, at least in a very concrete form. A half-semester is ample time to cover the ground. The interest shown in

mathematics from this simple study is gratifying, the improvement in geometrical imagination is valuable, and the idea of the mathematical study of essential structure is one that will go a long way in placing the correct estimate on the necessity of mathematics in a liberal education. If such a course is given by a mathematician rather than a mineralogist, and its bearing on the various problems in mathematics that it opens up is fully brought out, there are few subjects more stimulating to the student.

The text is based upon the "law of Bravais"—that crystalline structure is a finitely periodic reticulation. This law includes the law of rational indices. The first part is devoted to the study of the individual crystal, the second to the complex crystalline structures. There are two sections in the first part, the first of them being on the geometry of the crystal, the second on the physics of the crystal. In the latter the properties are classified as those due to discontinuous vectors, those due to continuous vectors. The treatment as a whole is clear and well put. The text is elementary.

JAMES BYRNIE SHAW.

Nouvelles Tables Trigonométriques Fondamentales. By H. ANDOYER. Paris, A. Hermann. 4to. xxxii + 602 pp.

IT is not easy to arouse much enthusiasm over a table of logarithms. Most of those who use aids to calculation are apt to regard them as tools necessary to the workshop, which are taken up and laid down with absolute indifference and with absolute confidence in their complete accuracy. Perhaps the latter is justified as far as the ordinary four-, five- and seven-place tables are concerned, and if this new issue were one of the many reprints that appear almost annually it would scarcely call for notice in the pages of the BULLETIN.

But there are features of M. Andoyer's work which merit special mention. The main portion of the work consists of the logarithms of the sines, cosines, tangents and cotangents to *fourteen* places at intervals of ten seconds of arc for the whole quadrant. An additional table gives the logarithms of $\sin\alpha/\alpha$, $\tan\alpha/\alpha$ for the same interval and for the first three degrees. Four further tables used in the calculations are printed: a page of logarithms to eighteen places of certain numbers; a page of formulas for the expansions of $\log\cos\frac{1}{2}\pi x - \log(1-x^2)$ in powers of x^2 as far as x^{40}, and similar

expansions for the sine and tangent; the logarithms of the trigonometric functions for every hundredth part of the quadrant to seventeen places, together with their variations as far as the latter are sensible; and finally the logarithms of the functions to fifteen places at nine-minute intervals, together with their variations per 10″ to the seventh order in the case of the cosines.

In undertaking the work, M. Andoyer had two courses open to him. One was the usual plan of getting the results from other tables by a more or less sustained effort at correction, interpolating where the older tables were not sufficiently subdivided. As he tells in his preface, there are only two original tables which aim at the degree of accuracy he wished to attain. The first is the Trigonometria Britannica of Henry Briggs published in 1633, the other the Tables du Cadastre computed between 1794 and 1799 under the direction of de Prony but never issued in printed form. Vlacq's well known tables were also published in 1633 but were only carried to ten places. All the tables of later date are founded on these. The tables of Briggs can be relied on to a unit in the thirteenth place, those of de Proney to the twelfth place.

The accuracy of observations has so much increased during the last fifty years that the common seven-place tables no longer satisfy the demands of those who have engaged in refined work, especially in astronomy. The problem is not so much that of getting the numerical value of a single function or of a few functions: in such cases one can usually adopt devices which grind out the result at the cost of trouble and time. Many of the present day problems are on a large scale. The calculations are turned over to professional computers, and special devices which a mathematician can adopt and use are frequently not well adapted for the computer who is only familiar with the ordinary methods. Extended tables and, if possible, mechanical devices are more and more sought after in order to economize time and money in scientific work, just as in business.

For various reasons, M. Andoyer decided to abandon the attempt to correct and extend the previous tables: there was no other course open but that of computing all the functions from the beginning. To anyone familiar with numerical work this seems an enormous task. One is astonished to read that the whole of it was done in less than two years without as-

sistance! I quote from the Preface: "Les calculs nécessaires pour l'établissement des présentes tables (sauf la Table I qui n'est pas trigonométrique) ont été faits entièrement à nouveau, par moi seul, sans aucun auxiliaire, même mécanique. Un travail régulier de chaque jour, et quelque goût naturel pour les calculs numériques m'ont permis, malgré mes occupations ordinaires et malgré quelques périodes d'inactivité ou de tâtonnements, de mener à bien ma tâche sans ennui en un temps suffisamment court, de juillet 1908 à mars 1910, soit un an et huit mois." Is there a single college or university professor in this country whose ordinary duties would leave him the amount of free time necessary to carry through in twenty months a series of calculations, the results of which occupy 600 quarto pages?

Even if M. Andoyer, whose publications in the line of celestial mechanics are well known, had occupied himself solely with the tables, the fact of their completion in so brief a time would have been worthy of notice. An examination of his Introduction reveals the methods he adopted for the purpose of abbreviating the computations. A study of these, too long to explain in a review, will repay well anyone who aspires to undertake the formation of similar tables. But it is in the methods adopted to test the accuracy of the final numbers that M. Andoyer has given the best proofs. He is not content with the common method of forming differences but has made actual use of such formulas as

$$\cos\alpha \cdot \cos\left(\alpha + \frac{\pi}{n}\right) \cdots \cos\left(\alpha + \frac{n-1}{n}\pi\right) = \pm\frac{\cos n\alpha}{2^{n-1}}$$

(where $n = 4n' \pm 1$) to test the numerical values of the functions of angles separated by considerable intervals. One may without danger agree with him when he affirms that the original manuscript is absolutely free from errors. The same care has been taken with the proofs, which have been read after the final impression; the list of errata at the end contains one error—a unit in the last place of one function,—the rest being merely defects of printing and errors amongst the signs indicating the magnitude of the remainder following the fourteenth place of decimals.

If a criticism is to be made, it is in the use of the sign just mentioned. M. Andoyer, following T. N. Thiele, puts a

+ after a number when the remainder lies between a quarter and three quarters of the last unit set down. This compels anyone who wishes to use the table to insert the sign or to provide for a possible error of three quarters of a unit in his last place. The ordinary method with no sign inserted leads to a maximum error of half a unit. It seems to the reviewer to be preferable to use two signs, a plus, for example, when the remainder is between a quarter and a half, and a minus with the last figure raised one unit when the remainder is between a half and three quarters. These signs can then be dropped without adjustment and with the usual error of half a unit.

This is, however, a very small matter. The author is to be greatly congratulated on the successful completion of his task, and on producing a volume which will undoubtedly be the standard for all future tables of the logarithms of trigonometric functions. The excellent typography—an important point—should not be forgotten. The publication has been made with aid of a subvention from the Fondation Commercy by the University of Paris.

ERNEST W. BROWN.

Annuaire du Bureau des Longitudes pour l'An 1912. Paris, Gauthier-Villars.

THE present issue has its usual share of improvements and additions by the alteration to more modern data of some of the astronomical and physical constants and the inclusion of new matter. For a handy volume of reference in such matters, it is probably unsurpassed by any other of the same size and cost. But the reviewer has had cause to wonder whether some portions of the matter are really of much value. This arose from his attempt to use the table of elements of the asteroids. On account of the convenient form in which they are given, he incautiously adopted the elements for about twenty asteroids, not for statistical purposes but for individual examinations. Two of them seemed rather remarkable and the Berlin Astronomische Jahrbuch was searched for previous values. It then appeared that the periods given were erroneous; No. 318 was set down with a period of 2204 days instead of 2104 days, and No. 624 with 4229 instead of 4429 days. These are of course slips in proof-reading, but none the less disconcerting. In tables of continuous functions such errors are easily seen and corrected; one cannot do this with isolated physical constants.

The appendices contain an account by M. G. Bigourdan of the mean temperature in different parts of France. The subject is illuminated by numerous diagrams showing the mean temperatures, daily, monthly, annually, winter, etc. M. P. Hatt gives an elementary account of the method of least squares illustrated by numerical examples. One of the best features is the full index, which enables the reader to find anything desired without trouble. For the first time the phenomena are given in legal time, that is, Greenwich time, which is nine minutes, twenty-one seconds behind Paris mean time. The change is due to a law passed on March 9, 1911, a convenience for which the astronomer as well as the traveller is grateful.

ERNEST W. BROWN.

NOTES.

THE nineteenth summer meeting of the AMERICAN MATHEMATICAL SOCIETY will be held at the University of Pennsylvania on Tuesday and Wednesday, September 10–11, 1912.

THE March number (volume 13, number 3) of the *Annals of Mathematics* contains the following papers: A third generalization of the groups of the regular polyhedrons," by G. A. MILLER; "A type of homogeneous linear differential equation," by L. A. HOWLAND; "On the complete logarithmic solution of the cubic equation," by R. E. GLEASON; "The circular numbers for a plane curve," by H. T. BURGESS; "On the sum of a certain triple series," by E. W. BROWN; "A theorem in difference equations on the alternations of nodes of linearly independent solutions," by E. J. MOULTON; "Periodic quadratic transformations in the plane," by VIRGIL SNYDER; "On the reduction of a system of linear differential forms of any order," by A. DRESDEN; "On the functional equation for the sine. Additional note," by E. B. VAN VLECK.

AT the meeting of the London mathematical society held on February 8 the following papers were read: By A. C. Dixon, "Exceptions to extensions of a theorem of Jacobi's"; by W. Burnside, "Some properties of groups whose orders are powers of primes"; by G. H. Hardy and J. E. Littlewood, "Some results concerning diophantine approximations."

AT the meeting of the Edinburgh Mathematical Society on March 8, the following papers were read: By G. D. C. Stokes, "An analytical study of plane rolling mechanisms"; by C. Tweedie, "The fundamental formula for the area of a triangle in analytical geometry"; by E. M. Horsburgh, "Proof of a fundamental relation between the bending moment and load curves"; by H. Levy, "An extension of Clairaut's differential equation."

THE annual meeting of the British association for the advancement of science will be held at Dundee, Scotland, September 4–10, under the presidency of Professor E. A. Schäfer. Professor H. L. CALLENDAR is chairman of section A, mathematical and physical sciences.

THE thirty-ninth summer meeting of the French association for the advancement of science will be held at Toulouse, August 1–7, under the presidency of Professor Gariel. Professor E. BELOT is chairman of the section of pure and applied mathematics.

IN response to the wish expressed at the Milan meeting of the international commission on the teaching of mathematics for a supply depot in which all the publications of the commission can be purchased, the firm of Georg et Cie, Geneva, Switzerland, has undertaken to provide such a repository, and to keep in stock a sufficient supply to meet all regular demands. The report of the Milan meeting was published in the *Enseignement Mathématique*, November 11, 1911. It contains a list of all the publications of the commission which had appeared at that time, together with their prices. This arrangement, and the provision for receiving the reports of the American subcommission from the Bureau of Education in Washington furnish an excellent opportunity to prepare for an intelligent understanding of the full report to be laid before the fifth international congress of mathematicians in Cambridge next August.

REPORTS of American committees XI, "Mathematics at West Point and Annapolis," and VI, "Mathematics in the technical secondary schools in the United States" of the International commission on the teaching of mathematics have just been issued by the United States bureau of edu-

cation in Washington. Copies of these and the preceding reports may be secured gratis on application.

THE twelfth regular meeting of the Rochester section of the association of teachers of mathematics in the Middle States and Maryland was held at Buffalo, March 9, 1912. The following papers were read: "Ways in which teachers of mathematics may grow," by C. N. Millard, with formal discussion by E. C. Brown and G. G. McEwen; "Notes on teaching radicals," by Miss M. M. Wardwell, with formal discussion by Miss M. E. Crofts and Miss J. C. Davis; "Suggestions on graphs in elementary algebra," by W. Betz; "Graphs in advanced secondary mathematics," by A. S. Gale. The association has other sectional organizations in New York, Philadelphia, and Syracuse, in all of which regular meetings are held.

THE firm of B. G. Teubner, in Leipzig and Berlin, has just issued a 64 page catalogue of mathematical models prepared by H. Wiener and P. Treutlein. The first part consists of 106 wire models representing quadric surfaces, helicoids, space curves, and linkages. The second part enumerates an extensive collection of wire frames and wooden models for use in solid geometry and in elementary projective geometry.

THE following university courses in mathematics are announced for the summer semester, April 15 to August 15:

UNIVERSITY OF MUNICH. — By Professor F. LINDEMANN: Integral calculus, five hours; Theory of linear differential equations and conformal representation, four hours; Quadrature of the circle, two hours; Seminar, two hours. — By Professor A. VOSS: Analytic geometry of space, four hours; Analytic mechanics, II, four hours; Seminar, two hours. — By Professor A. PRINGSHEIM: Elementary theory of ordinary differential equations, four hours; Analytic theory of numbers, four hours. — By Professor H. BRUNN: Analytic geometry and analysis situs, two hours. — By Professor K. DOEHLEMANN: Descriptive geometry, II, with exercises, six hours; Synthetic geometry, II, four hours; Imaginary elements in geometry, one hour. — By Professor G. HARTOGS: Differential calculus, five hours. — By Dr. K. BÖHM: Selected chapters of differential geometry, three hours; Theory of probabilities with application to insurance, three hours.

UNIVERSITY OF STRASSBURG.—By Professor H. WEBER: Definite integrals and introduction to the theory of functions, four hours; Algebra, two hours; Seminar, two hours.—By Professor F. SCHUR: Theoretical mechanics, four hours; Ordinary differential equations, two hours; Seminar, two hours.—By Professor M. SIMON: Stereometry of non-euclidean geometry, one hour.—By Professor J. WELLSTEIN: Elliptic and hyperelliptic functions, four hours; Theory of invariants, two hours; Seminar, two hours.—By Professor R. v. MISES: Hydromechanics, three hours; Differential and integral equations of mechanics, three hours; Seminar, two hours.—By Dr. P. EPSTEIN: Introduction to higher mathematics, two hours; Seminar, two hours.—By Dr. A. SPEISER: Theory of algebraic number fields, two hours; Recent problems in the theory of groups, two hours; Proseminar, two hours.

DR. M. RIESZ has been appointed docent in mathematics at the University of Stockholm.

DR. SLESZYNSKI has been appointed docent in mathematics at the University of Cracow.

DR. V. POLLACK, of the technical school at Vienna, has been promoted to an associate professorship of geodesy.

DR. E. T. WHITTAKER, royal astronomer of Ireland, has been appointed professor of mathematics at the University of Edinburgh, as successor of the late Professor G. Chrystal.

AT Purdue University Dr. E. G. BILL has been promoted to an assistant professorship of mathematics.

RECENT catalogues of second-hand books:—A. Hermann et Fils, 6 rue de la Sorbonne, Paris, catalogue 113, 1900 titles in mathematics, physics, and mechanics.—W. Heffer and Sons, Cambridge, England, catalogue 88, 2100 titles in mathematics and science.—Oswald Weigel, Königstrasse 1, Leipzig, catalogue 25, 600 titles in exact sciences.—Joseph Baer und Co., Hochstrasse 6, Frankfort, catalogue 599, periodicals 1700 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

D'ADHÉMAR (R.). See MONTESSUS (R. DE).

BALL (W. W. R.). Mathematical recreations and essays. 5th edition. New York, Macmillan, 1911. 12mo. 16+506 pp. $2.75

BLAUERT (M.). Ueber einige Anwendungen der elliptischen Funktionen auf die Theorie des ebenen Gelenkvierecks. (Diss.) Rostock, 1911.

BOREL (E.). See TANNERY (J.).

BRAUDE (F.). Ueber einige Verallgemeinerungen des Begriffes der Mannheimschen Kurve. (Diss.) Heidelberg, 1911.

CARL (A.). Ueber höhere Rückkehr- und Wendepole. (Diss.) Jena, 1911.

DEL CARRAL (J. I.). Nuevos metodos para resolver ecuaciones numericas. Madrid, Romo, 1911. 8vo. 303 pp.

DIOPHANTE. Le traité des nombres polygones de Diophante d'Alexandrie. Traduction française avec une introduction, par G. Massoutié. Paris, Gauthier-Villars, 1911. 8vo. 32 pp. Fr. 1.50

DÜSING (K.). Die Elemente der Differential- und Integralrechnung in geometrischer Methode. Ausgabe B. Hannover, Jänicke, 1911.

EGGERS (G.). Ueber gewisse mit den Kegelschnitten zusammenhängende Kurven höherer Ordnung. (Diss.). Halle, 1911.

FAIRONS (J.). Sur les involutions du quatrième ordre. Bruxelles, Hayez, 1909. 8vo. 68 pp. Fr. 1.75

FENDER (W.). Zur Theorie von verallgemeinerten Bernouillischen und Eulerschen Zahlen. (Diss.) Jena, 1911.

FRENZEL (C.). Die Fundamente für eine elementare Einleitung in die Differential- und Integralrechnung. (Festschrift zur 50jährigen Jubelfeier des Gymnasiums zu Lauenburg. 2ter Teil.) Lauenburg, 1911. 8vo. 47 pp.

GALDEANO (Z. DE). Algunos conceptos fundamentales en un curso de análisis matemàtico y de las funciones. Zaragoza, Casañal, 1911. P. 2.50

GRÜNBAUM (H.). Funktionenlehre und Elemente der Differential- und Integralrechnung. 3te umgearbeitete und vermehrte Auflage des Lehr- und Übungsbuches der Differentialrechnung. Stuttgart, Grub, 1912. 8vo. 12+196 pp. M. 4.00

HELMHOLTZ (H. v.) Volksausgabe in einem Bande. Von L. Königsberger. Braunschweig, Vieweg, 1911. 14+356 pp. M. 4.50

HOPPE (E.). Mathematik und Astronomie im klassischen Alterthum. Heidelberg, Winter, 1911. M. 7.00

INNES (R.). A logical notation for mathematics. Kapstadt, 1911.

JASCHEWSKI (J.). See MINKOWSKI (H.).

JUNGE (G.). Ueber den Fehler bei logarithmischen Rechnungen. (Progr.) 1ter Teil. Landsberg, 1911. 19 pp.

KNESER (A.). Mathematik und Natur. (Rektorats-Rede, Breslau Universität.) Breslau, 1911.

KÖNIGSBERGER (L.). See HELMHOLTZ (H. v.).

KÖSSLER (H.). Über windschiefe Kegelschnitte. (Diss.) Halle, 1911.

LORIA (G.). Poliedri, curve e superficie secondo i metodi della geometria descrittiva. Milano, Hoepli, 1912.

MASSOUTIÉ (G.). See DIOPHANTE.

MINKOWSKI (H.). Raum und Zeit. (Russian translation by J. Jaschewski.) St. Petersburg, 1911.

MONTESSUS (R. DE) e D'ADHÉMAR (R.). Calcul numérique. 1re partie: Operations arithmétiques et algébriques. 2e partie: Intégration. (Encyclopédie scientifique.) Paris, Doin, 1911. Fr. 5.00

PLEMELJ (J.). Potentialtheoretische Untersuchungen. (Gekrönte Preisschrift.) Leipzig, Teubner, 1911. 8vo. 19+100 pp. M. 6.00

REIMERDES (O.). Die Niveau- und Fallinien auf Flächen, insbesondere auf Modulflächen analytischer Funktionen. (Diss.) Kiel, 1911.

RIEDER (K). Polynomische Entwicklungen von Funktionen einer komplexen Variablen. (Diss.) Basel, 1911.

RIEMANN (B.). See WEBER (H.).

TANNERY (J.). Science et philosophie. Avec une notice de E. Borel. Paris, Alcan, 1911. 16mo. 16+336 pp. Fr. 3.50

TIKHOMANDRITZKY (M.). Eléments de la théorie des intégrales abéliennes. Nouvelle édition, revue, corrigée, complétée de notes et en partie refaite entiérement. St. Petersburg, 1911.

WEBER (H.). Die partiellen Differentialgleichungen der mathematischen Physik. Nach Riemanns Vorlesungen. 5te Auflage. 2ter Band. Braunschweig, Vieweg, 1912. 8vo. 575 pp. M. 16.80

WILSON (E. B.). Advanced calculus. A text upon select parts of differential calculus, differential equations, integral calculus, theory of functions, with numerous exercises. Boston, Ginn, 1911. 8vo. 566 pp. Cloth. $2.75

II. ELEMENTARY MATHEMATICS.

BAKER (W. M.) and BOURNE (A. A.). A new geometry. Books 1–3. London, Bell, 1912. 8vo. 148 pp. 1s. 6d.

BENDT (F.). Grundzüge der Trigonometrie. 4te Auflage. Leipzig, Weber, 1911. 8+135 pp. Cloth. M. 2.00

BODE (P.). Die Reform des mathematischen und naturwissenschaftlichen Unterrichts an höheren Schulen in der Gegenwart. (Diss.) Leipzig, 1911.

BORCHARDT (W. G.) and PEROTT (A. D.). Geometry for schools. Volume 3, stage 3, section 2. London, Bell, 1912. 8vo. 1s.

BOURNE (A. A.). See BAKER (W. M.).

BRUNO (G. M.). Geometria, con numerosos ejercicios. 2e edición, revisada y aumentada. Tours, Mame, 1911. 16mo. 392 pp.

DIECKMANN. Leitfaden zur Aufgabensammlung für den Unterricht in Algebra. Leipzig, Degener, 1911. M. 1.50

DONADT (A.). Arithmetik und Algebra zum Selbstunterricht. Leipzig, Brandstetter, 1911. M. 4.50

EVERETT (J. P.). See STEIMLE (C. P.).

EXERCICES de géométrie comprenant l'exposé des méthodes géométriques et 2000 questions résolues, par F. G. M. 5e édition. Paris, de Gigord, 1912. 8vo. 24+1302 pp.

GERMEAU (J.). Essai d'un cours de trigonométrie rectiligne avec de nombreuses applications. 2e édition. Namur, Wesmael-Charlier, 1911. 8vo. 98 pp. Fr. 1.50

GIRNDT (M.). Leitfaden der bautechnischen Algebra. 4te Auflage. Leipzig, Teubner, 1911. M. 1.50

—— und LIEBMANN (A.). Logarithmen und Kurventabellen für den Gebrauch an Tiefbauschulen. Leipzig, 1911. M. 1.20

GODFREY (C.) and SIDDONS (A. W.). A shorter geometry. London, Cambridge University Press, 1912. 8vo. 324 pp. 2s. 6d.

GUERBY (E.). See ROUSIER (G.).

GUICHARD (C.). Traité de géométrie. Tome 1er. 4e édition. Paris, Vuibert, 1911. 8vo. 8+566 pp.

HOCEVAR (F.). Lehr- und Übungsbuch der Geometrie für Gymnasien und Realgymnasien. 7te Auflage. Wien, Tempsky, 1912. Kr. 2.80

HOCHHEIM (A.). Aufgaben aus der analytischen Geometrie der Ebene. Heft 1: Die gerade Linie, der Punkt, der Kreis. A: Aufgaben. B: Auflösungen. 4te, vermehrte Auflage, bearbeitet von O. Jahn und F. Hochheim. Leipzig, Teubner, 1911. Cloth. M. 5.00

——. Aufgaben aus der analytischen Geometrie der Ebene. Heft 3: Die Kegelschnitte. Abteilung II. A: Aufgaben. B: Auflösungen. 2te, vermehrte Auflage, bearbeitet von F. Hochheim. Leipzig, Teubner, 1911. Cloth. M. 4.20

ISVOLSKY (N.). Treatise on geometry. I: Plane geometry. II: Solid geometry. (Russian.) Moscow, 1911. 266+127 pp.

KOMMERELL (V.) und KOMMERELL (R.). Analytische Geometrie. Für den Schulgebrauch bearbeitet. Tübingen, Laupp, 1912. 8vo. 8+180 pp. M. 2.40

LIEBMANN (A.). See GIRNDT (M.).

LIETZMANN (W.). Der mathematische Elementarunterricht und die Mathematik an den Lehrerbildungsanstalten. (5ter Band der Abhandlungen über den mathematischen Unterricht in Deutschland.) 1tes Heft: Stoff und Methode des Rechnenunterrichts in Deutschland. Leipzig, Teubner, 1912. 8vo. 7+125 pp. M. 3.00

LÖRCHER (O.). Kurzgefasster methodischer Lehrgang der ebenen Geometrie nebst Grundzügen der Raumgeometrie. Stuttgart, Grub, 1912. 8vo. 8+72 pp. M. 1.20

MAINGIE (J.). Manuel d'algèbre élémentaire. Namur, Wasmael-Charlier, 1911. 16mo. 79 pp. Fr. 1.25

MATRICULATION mathematics papers. University of London. 1905–1911. London, Clive, 1912. 8vo. 112 pp. 1s. 6d.

MELDAU (H.). See SCHILLING (C.).

MILLIS (C. T.). Technical arithmetic and geometry. 2nd edition, revised. London, Methuen, 1912. 8vo. 316 pp. 3s. 6d.

MONNET (G.). See NAUD (L.).

NAUD (L.) et MONNET (G.). Douze cents problèmes à l'usage des candidats aux examens de l'administration des postes et des télégraphes. Paris, Palais des Sociétés Savantes, 1911. 8vo. 96 pp.

RODGER (J.). An introduction to the use of common logarithms. London, Blackie, 1912. 8vo. Limp. 1s.

ROUSIER (G.) et GUERBY (E.). Cours de trigonométrie, à l'usage des élèves des classes de mathématiques et de première C et D. Paris, Hatier, 1912. 12mo. 232 pp.

SALOMON (A.). Leçons de géométrie à l'usage de l'enseignement secondaire des jeunes filles. Géométrie plane. 5e édition. Paris, Vuibert, 1911. 16mo. 230 pp.

SCHILLING (C.) und MELDAU (H.). Der mathematische Unterricht an den deutschen Navigationsschulen. (4ter Band, 4tes Heft der Abhandlungen über den mathematischen Unterricht in Deutschland.) Leipzig, Teubner, 1912. 8vo. 6+82 pp. M. 2.00

SCHÜLKE (A.). Aufgabensammlung aus der reinen und angewandten Mathematik. 1ter Band. 2te Auflage. Leipzig, Teubner, 1912. M. 2.20

SCOTT (E. E.). Tables of logarithms and anti-logarithms to five places. Student's edition. London, Layton, 1912. 8vo. 390 pp. 5s.

SIDDONS (A. W.). See GODFREY (C.).

STEIMLE (C. P.) and EVERETT (J. P.). Key to Lyman's plane and solid geometry. New York, American Book Co., 1912. 12mo. 128 pp. Half leather. $1.00

SUPPANTSCHITSCH (R.). Lehrbuch der Arithmetik und Algebra. Wien, Tempsky, 1912.

——. Trigonometrie und analytische Geometrie. Wien, Tempsky, 1912. M. 4.40

VEDDER. Rechenergebnisse zu Dieckmann, Algebra. Leipzig, Degener, 1911. M. 2.00

VESTRUM. Der Begriff "Richtung," seine Stellung und Bedeutung in der elementaren Geometrie. Kristiania. Cammermeyer, 1911.

WILK (H.). Arithmetik und Algebra für höhere Mädchenschulen. 2ter Teil. Dresden, Bleyl und Kaemmerer, 1911. M. 1.50

III. APPLIED MATHEMATICS.

BESANT (W. H.) and RAMSAY (A. S.). A treatise on hydromechanics. Part I: Hydrostatics. 7th edition. London, Bell, 1911. 8vo. 275 pp.

BLANCARNOUX (P.). Toute la mécanique rationelle et appliquée à la portée de tous. Panorama méthodique et complet en 12 volumes. Tome 7: Appareils auxiliaires. Paris, Geisler, 1912. 8vo. 132 pp.

BOUSSINESQ. See MENNERET.

BOYD (J. E.). Strength of materials. New York, McGraw-Hill, 1911. 8vo. 12+295 pp. $2.50

CORNUT, TEXIER et DUTHEIL. Notions élémentaires de mécanique générale, à l'usage des écoles primaires supérieures. 3e année: Mécanique appliquée aux machines. Paris, Nathan, 1912. 8vo. 258 pp.

DUTHEIL. See CORNUT.

ENZYKLOPÄDIE der mathematischen Wissenschaften. Band IV: Mechanik. 1 II, Heft 2: Dynamische Probleme der Maschinenlehre, von R. v. Mises. Leipzig, Teubner, 1911. 8vo.

—— der mathematischen Wissenschaften. Band IV: Mechanik. 2 II, Heft 6: Begriffliche Grundlagen der statistischen Auffassung in der Mechanik, von P. U. T. Ehrenfest. Leipzig, Teubner, 1912.

FOERSTER (M.). Taschenbuch für Bauingenieure. Berlin, Springer, 1911. 8vo. 15+1912 pp.

FOURQUET (J.). Eléments de géométrie descriptive appliquée au dessin et à la construction. Tome 3: Chaudronnerie, zinguerie, ferblanterie. Lille, Janny, 1912. 8vo. 164 pp.

GASTOU (R.). La théorie de l'aviation. Son application à l'aéroplane. Paris, Vivien, 1911. 8vo. 31 pp. Fr. 1.50

HANCOCK (E. L.). See SLOCUM (S. E.).

HARTMANN (F. M.). Heat and thermodynamics. New York, McGraw-Hill, 1911. 8vo. 7+346 pp. $3.00

IHERING (A. v.). Die Mechanik der flüssigen Körper. (Aus Natur und Geisteswelt, Band 304.) Leipzig, Teubner, 1912.

INGRAM (E. L.). Geodetic surveying and the adjustment of observations (method of least squares). New York, McGraw-Hill, 1911. 8vo. 20+389 pp. $3.00

JACOBY (H. S.). See MERRIMAN (M.).

KENNELLY (A. E.). The application of hyperbolic functions to electrical engineering problems. London, Hodder, 1912. 8vo. 288 pp. 6s.

KÖLSCH (O.). Gleichgang und Massenkräfte bei Fahr- und Flugzeugmaschinen. Berlin, Springer, 1911. M. 5.00

MENNERET et BOUSSINESQ. Mouvement oscillatoire et mouvement uniforme des liquides dans les tubes cylindriques. Frottement interne. Aperçu théorique sur les oscillations d'une colonne liquide dans un tube en U. Tours, Deslis, 1911. 8vo. 28 pp.

MERRIMAN (M.). Treatise on hydraulics. 9th edition, revised and reset. London, Chapman and Hall, 1912. 8vo. 17s.

—— and JACOBY (H. S.). A textbook on roofs and bridges. Part 3: Bridge design. 5th edition. London, Chapman and Hall, 1912. 8vo. 10s. 6d.

POINCARÉ (H.). Leçons sur les hypothèses cosmogoniques professées à la Sorbonne. Rédigées par H. Vergne. Paris, Hermann, 1911. 8vo. 25+296 pp.

PUGEHL (F.). Ueber ein von F. Klein gestelltes Problem aus der Theorie der Bewegung eines starren Körpers. (Diss.) Königsberg, 1911.

RAMSAY (A. S.). See BESANT (W. H.).

SCHAU (A.). Statik. Leitfaden für den Unterricht an Baugewerbeschulen. 1ter Teil. Leipzig, Teubner, 1911. M. 2.40

SCHINDLER (R.). Die Mechanik des Mondes. Luzern, 1911.

SLOCUM (S. E.) and HANCOCK (E. L.). Textbook on the strength of materials. Revised edition. Boston, Ginn, 1911. 8vo. 36+372 pp.

STEINMETZ (C. P.). Elementary lectures on electric discharges, waves and impulses, and other transients. New York, McGraw-Hill, 1911. 8vo. 11+149 pp. $2.00

TEXIER. See CORNUT.

VERGNE (H.). See POINCARÉ (H.).